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ENTRAINMENT AND EXPANSION CONTROLLED FIREBALL RISE

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6 ENTRAINMENT AND EXPANSION CONTROLLED
FIREBALL RISE

10 by Gerhard Rohringer

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SUMMARY

This ~~The present~~ paper reports on a detailed analysis of the buoyant rise of fireballs in the earth's atmosphere. Formulae for the rise velocity and height, and the density, mass, radius and expansion velocity of the fireball are given. The computation of fireball temperature is discussed in detail; no explicit expression could be given because of the nature of the problem. The assumptions and simplifications on which the analysis rests are summarized in a separate section. In order to facilitate applications, a complete numerical example is given. Frequently used quantities are calculated and presented in graphical and tabular form.

ACKNOWLEDGEMENTS

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LIST OF SYMBOLS

The page where the symbol first occurs is given in parentheses.

A:	cross sectional area of the fireball (p. 10)
C_d:	drag coefficient (p. 10, Appendix 2)
dV_e:	change in total fireball volume due to entrainment (p. 12)
dV_{ex}:	change in total fireball volume due to expansion (p. 10)
g:	gravitational acceleration ~ 10^3 cm sec^{-2} (p. 10)
$\overline{\gamma}_F$:	average γ of the fireball gas (p. 11, Appendix 4)
h:	height to which the fireball has risen during the expansion controlled phase of the fireball rise (p. 26)
$h_{tot,a}$:	total height achieved during the expansion controlled phase of the fireball rise (p. 26)
$h_{tot,N}$:	the height corresponding to a rise of N instantaneous radii during the entrainment controlled phase of the fireball rise (p. 18)
$H_{S\rho}$:	scale height of atmospheric density distribution (p. 11, Appendix 5)
H_{ST}:	scale height of the atmospheric temperature distribution (p. 11, Appendix 5)
H_{Se}:	equivalent scale height (p. 11, Appendix 5)
λ:	mass entrainment factor (p. 12)
M_0:	initial mass of the fireball (p. 14)
$(M/M_0)_N$:	ratio of the total fireball mass to the initial fireball mass after the fireball has risen N instantaneous radii during the entrainment controlled portion of its rise (p. 19)

N:	number of instantaneous radii the fireball has risen during the entrainment controlled phase of its rise (p.17)
R_A:	specific gas constant of the ambient air (p.35)
$R_{F,i}$:	specific gas constant of the originally hot fireball gas (p.35)
$\overline{R}_F$:	specific average gas constant of the fireball gases (p.11)
$\overline{R}_{F,N}$:	specific average gas constant of the fireball gases after the fireball has risen N instantaneous radii (p.21)
r_F:	fireball radius (p.12)
$r_{F,k}$:	fireball radius at the beginning of the k^{th} step (p.17)
$r_{F,h}$:	radius of the fireball after it has risen a height h during the expansion controlled phase of its rise (p.27)
$r_{F,N}$:	radius of the fireball after it has risen N instantaneous radii during the entrainment controlled phase of its rise (p.19)
ρ_A:	ambient air density (p.10)
$\rho_{A,h}$:	ambient air density when the fireball has risen a height h during the expansion controlled phase of its rise (p.27)
$\rho_{A,N}$:	ambient air density after the fireball has risen N instantaneous radii during the entrainment controlled phase of its rise (p.19)
$\overline{\rho}_e(t_e, t)$:	spatial average at time t of the density of air entrained at time t_e (p.14)
$\rho_{e,i}$:	density of the entrained air immediately after entrainment (p.12)
$\overline{\rho}_F$:	average fireball density (p.10)
$\overline{\rho}_{F,h}$:	average fireball density after the fireball has risen a height h during the expansion controlled phase of its rise (p.27)
$\overline{\rho}_{F,i}(t)$:	spatial average density of the original, hot fireball gas at time t (p.14)

$\rho_{F,i}$:	density of the originally hot fireball gas (p. 35)
$\overline{\rho}_{F,N}$:	average fireball density after the fireball has risen N instantaneous radii during the entrainment controlled portion of its rise (p.19)
$(\overline{\rho}_F/\rho_A)_h$:	ratio of average fireball density to ambient air density after the fireball has risen a height h during the expansion controlled phase of its rise (p.26)
$(\overline{\rho}_F/\rho_A)_N$:	ratio of average fireball density to ambient air density after the fireball has risen N instantaneous radii during the entrainment controlled portion of its rise (p.19)
$(\overline{\rho}_F/\rho_A)_O$:	ratio of average fireball density to ambient air density before the entrainment controlled phase of the rise begins (p.18)
T_A:	temperature of the ambient air (p.11)
$T_{A,h}$:	temperature of the ambient air when the fireball has risen a height h during the expansion controlled phase of its rise (p. 28)
$T_{A,N}$:	temperature of the ambient air at the beginning of the N^{th} step (p.21)
T_E:	temperature at which the Rosseland mean free path increases again (p. 2)
$\overline{T}_F$:	average fireball temperature (p.11)
$\overline{T}_{F,h}$:	average temperature of the fireball after it has risen a height h during the expansion controlled phase of its rise (p.28)
$\overline{T}_{F,N}$:	average temperature of the fireball at the beginning of the N^{th} step during the entrainment controlled phase of its rise (p.21)
t_{acc}:	time after burst at which the analysis becomes applicable and initial values must be chosen (p. 16)
$t_{tot,N}$:	time corresponding to a rise of N instantaneous radii during the entrainment controlled phase of the rise (p.18)
v:	rise speed (p. 10)
V:	total fireball volume (p. 10)

INTRODUCTION

The solution of the problem of physically describing low-altitude fireballs is best undertaken in three steps. These are: the radiative growth of the fireball; the establishment of pressure equilibrium between the fireball and the ambient atmosphere; the buoyant rise of the fireball. Each of the steps is dominated by different physical phenomena and consequently by a different set of equations. Breaking the problem into distinct steps results therefore in uniformity of approach throughout each step. A further important simplification is the possibility of suitably averaging the detailed results of one step in order to obtain better manageable starting values for the next step. The mentioned steps are discussed in general in the following. The problem of buoyant fireball rise is treated in detail.

GENERAL DISCUSSION

1. Radiative Fireball Growth

The first step is the analysis of the initial radiative growth of the fireball. Immediately after deposition of the X-ray energy the fireball grows by reradiation. The process is described by the radiation diffusion equation. The important quantity appearing in this equation is the Rosseland mean free path. For a semiquantitative understanding of the radiation process it is necessary to know that the Rosseland mean free path decreases with temperature to a certain minimum at a temperature T_E, and then increases again. (The Rosseland mean free path also decreases with density. This is important for the establishment of an upper height limit to which the radiatively growing fireball is described by the radiation diffusion equation.) Plotting fireball temperature T normalized to the temperature T_F at the center of the fireball with respect to radial distance r normalized to fireball radius r_F, one obtains the schematic drawing shown in Figure 1. (The distance from the fireball center at which the temperature is close to ambient temperature T_A is defined as r_F.) Initially the temperature T_F in the center of the fireball is high so that the Rosseland mean free path is long and a large portion of the fireball is isothermal. At the fireball edge the temperature drops so that the Rosseland mean free path decreases and the fireball grows in size by radiation diffusion. The temperature T_F at the center drops as the radius r_F increases and more air is engulfed into the fireball.

Initially, when T_F is high, most of the radiation energy is transported at temperatures much exceeding T_E, that is above that temperature for which the mean free path of radiation is a minimum. (This can be seen most clearly by considering that for large T_F, T_E/T_F is well down on the T/T_F, r/r_F curve. Most of the radiation transport takes place to the left of this point, at temperatures exceeding T_E.) Only a small fraction of the total radiation energy escapes therefore to large distances. The fireball grows and cools by radiative expansion at the fireball edge where the temperature drops and the Rosseland mean free path decreases. As the temperature at the center of

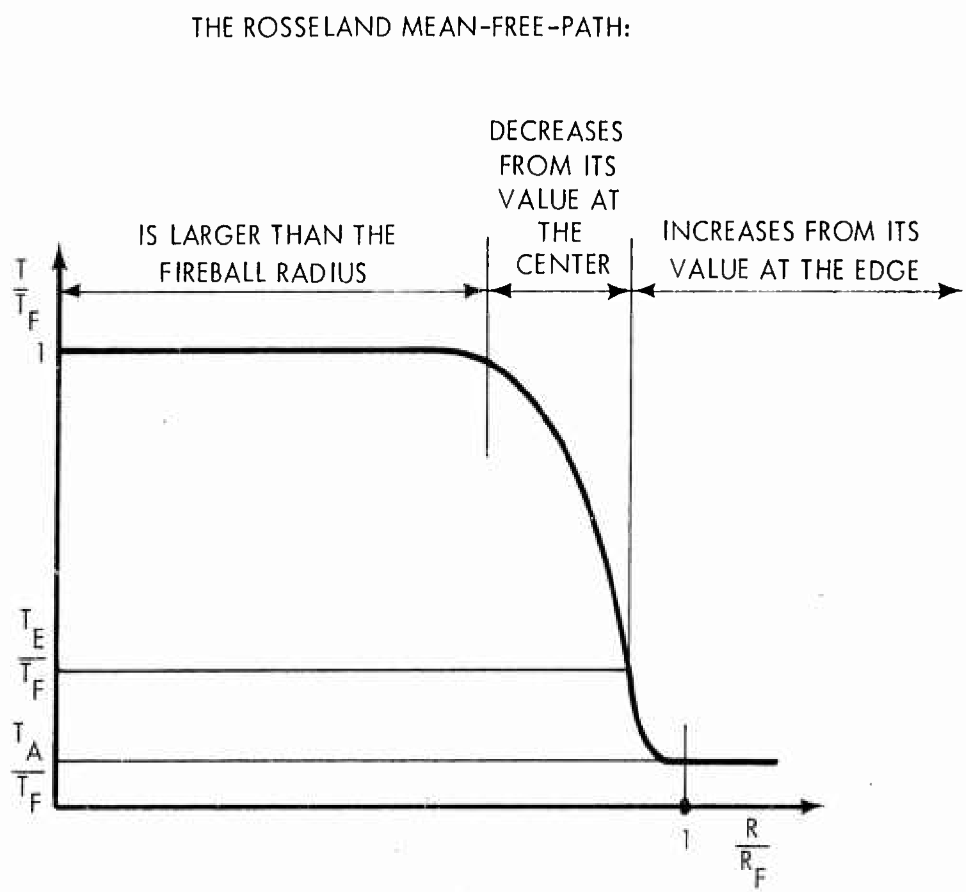

Figure 1. 1. Temperature Distribution Within the Fireball

the fireball drops, more of the radiative energy is transported at temperatures below T_E and can escape to large distances. (Again, this can be seen most easily by considering that for low T_F, T_E/T_F is up on the T/T_F, r/r_F curve and a large portion of the radiation transport occurs at temperatures below T_E.) An increasing fraction of the total radiation energy escapes to large distances. This is one of two processes terminating radiative fireball growth.

The other process is hydrodynamic expansion of the fireball. During the time of radiative fireball growth the high pressure within the fireball accelerates the air particles away from the center of the burst. Once the speed of hydrodynamic expansion, which is nearly equal to the speed of sound within the fireball, becomes larger than that of radiative expansion, the radiative expansion phase is terminated. A sizeable fraction of the fireball's heat content may be transferred into blast energy and carried away to large distances. Hydrodynamic expansion competes therefore with radiative energy transport to large distances in terminating the radiative growth of the fireball.

The computation of radiative fireball growth may be one-dimensional, that is spherically symmetric, as long as the fireball radius is small compared to a scale height. Otherwise a two-dimensional model should be used. The significant results of the computation are fireball radius and fireball temperature at the end of the radiative phase.

In case the radiative expansion phase is terminated by hydrodynamic expansion, the corresponding density, temperature and velocity distributions within the fireball would be of interest for a more detailed analysis of the problem.

2. Establishment of Pressure Equilibrium

The second step in the physical description of low-altitude fireballs involves the analysis of the processes taking place between the end of the radiative expansion phase and the establishment of an approximate pressure equilibrium between the fireball and the ambient atmosphere.

In case hydrodynamic expansion ended the radiative expansion phase, fireball cooling by both radiation and hydrodynamic expansion must be considered from the beginning. This is likely to be the case at low altitudes where the mean free path of radiation is small. It is important to realize that radiative cooling depends strongly upon density. The latter depends upon hydrodynamic expansion which in turn is a function of temperature. This interaction between cooling by radiation, cooling by the generation of a blast wave, and cooling by expansion of the fireball gas makes the second step a difficult one. It must be noted that the hydrodynamic expansion itself may cause the formation of a new fireball by shock heating the ambient air to high temperatures. This fireball will also cool by radiation and expansion. It is convenient to refer to the fireball produced by radiative expansion as the "fireball" proper. The fireball produced by the hydrodynamic shock is best referred to as the "shock-heated region."

In case the end of the radiative expansion phase was marked by the transmission of a significant fraction of the radiation energy to large distances, fireball cooling by radiation is computed first and hydrodynamic expansion is taken into account once it becomes significant. Since the mean free path of radiation increases with decreasing density, the termination of the radiative phase by radiative cooling is likely to occur at high altitudes. A rough measure for the time until significant hydrodynamic motion sets in is obtained as that time which the rarefaction wave needs to travel one-tenth of the fireball radius inward. By then, thirty percent of the fireball mass are affected by hydrodynamic motion.

During the time of fireball expansion to pressure equilibrium, a "pressure force" (see Reference 1) acts upon the fireball. This force is a result of the exponential density distribution in the atmosphere which causes the pressure at the bottom of the isothermal fireball to exceed that at its top. The resulting force will drive portions of the fireball upward until pressure equilibrium with the ambient atmosphere has been attained. The acceleration due to the pressure forces drops from a value of $g(R_F T_F / R_A T_A)$ at the beginning of the expansion phase to zero at its end. (The subscripts F refer to the fireball and the subscripts A to the ambient atmosphere. R is the specific gas constant.) At altitudes and yields where the fireball radii become comparable to about three-tenths of an atmospheric scale height, the total mass of ambient air above the burst is comparable or small compared to the mass of air within the fireball and the "pressure force" will be able to drive the fireball into a ballistic trajectory (see Reference 1). This means that the fireball speed will be affected only insignificantly by the entrainment of ambient air and by drag and mainly be slowed down by gravitational deceleration. In addition, the time until pressure equilibrium between fireball and ambient atmosphere is established increases with height for the same yield, so that the "pressure force" acts longer at high altitudes thus achieving higher speeds.

The "pressure force" is not the only force acting upon the fireball. As soon as the fireball density drops below the ambient value, a buoyancy force appears. This force in contrast to the "pressure force" continues to act after pressure equilibrium has been established. The acceleration of the fireball due to buoyancy increases from zero at the beginning to 2g at the end of the expansion phase. Thereafter it declines, but persists until the fireball density has become equal to the ambient density. At low altitudes, the large mass of air above the fireball prevents the fireball from following a ballistic trajectory. Its speed attained through the "pressure force" will rapidly be slowed down by entrainment and drag. The speed attained through the buoyancy force on the other hand persists for a long time and carries the fireball to higher altitudes. Even so, the "pressure force" may still influence the initial portion of the fireball rise significantly.

A one-dimensional, spherically symmetric calculation of the second step will show many of its essential features. In particular, it will give average values of fireball radius, temperature and density. A one-dimensional model may be satisfactory for small ratios of fireball radius to scale height. It will not bring out the effect of the pressure force and of buoyant effects. For significant ratios of fireball radius to scale height, a two-dimensional model is preferable. It will yield rise speeds due to the pressure as well as the buoyancy force.

3. Buoyant Rise

The third step in the physical description of low-altitude fireballs for those cases where it is more significant than rise due to the "pressure force" is the analysis of the buoyant rise. At the termination of the fireball's expansion to ambient pressure the fireball will be rising with some speed, which to a certain extent is due to the "pressure force" and to a certain extent to the buoyancy force. It will be shown in a later section that the rise speed is small enough so that the speed of fireball expansion is below the sound speed within the fireball. This allows the fireball to remain in pressure equilibrium with the ambient atmosphere.

Initially, cooling by radiation dominates. Eventually, entrainment of ambient air becomes important. The change in fireball temperature and density due to radiation and entrainment and also due to expansion alters the radiative fireball properties. This interaction between the various processes makes the analysis difficult. As the fireball temperature decreases by radiation its density increases and the buoyancy force diminishes.

Once radiative cooling is less important a mechanism for dissipation of the fireball's heat content than entrainment of ambient air, the latter will decrease the fireball's average temperature, increase its average density and thus reduce the buoyancy force. The details of the process will depend upon the degree of mixing and heat exchange occurring between the original hot fireball gas, the already entrained and heated ambient air and the newly entrained air. The fireball will eventually consist of a mixture of gases each at a different temperature and at a different density, but all at nearly equal pressure. This non-uniformity and the exponential pressure and density distribution (see Reference 1) within the fireball will set up internal motions. Some of these will be random in nature because they are caused by the particular way in which portions of air are entrained, mixed and heated. They are the cause of turbulence within the fireball. Other motions, most likely those due to the exponential character of the atmosphere, will be independent of entrainment. They may cause predictable effects. The formation of the torus or smoke ring (see Reference 1) seems to be the most important example of this kind of motion.

Expansion during buoyant rise will also decrease fireball temperature and increase its density. It can be shown that this effect will eventually be more important than mass entrainment. It always controls the final phase of buoyant fireball rise.

In the following, a simplified analysis of the mass entrainment and expansion controlled phase of buoyant fireball rise is presented.

ANALYSIS

4. Assumptions and Simplifications for the Analysis of Entrainment and Expansion-Controlled Buoyant Rise

In the present report, a much simplified solution for the buoyant fireball rise is presented. Its basic assumptions and simplifications are:

It is assumed that the fireball is of spherical shape throughout the duration of the buoyant rise. Its physical description will be in terms of average values of density, pressure and temperature. The assumption of a spherical shape is not realistic in view of the fact that all buoyantly-rising fireballs eventually change from a spherical into a toroidal shape. The calculation of only average values may justify the assumption of a spherical shape.

The aerodynamic drag of the rising fireball is assumed to be proportional to its cross-sectional area, the square of its speed, the density of the ambient atmosphere and a constant drag coefficient. For the numerical example presented, the drag coefficient of a solid sphere is assumed. While it is difficult to assess the applicability of the drag coefficient of a sphere, the general assumption of aerodynamic drag seems to be reasonable. All buoyantly-rising fireballs are much smaller than an atmospheric scale height. They do therefore move in the atmosphere under the same conditions as those bodies do, for which aerodynamic drag has been found to apply in moving through a gas of uniform density. It also turns out that the drag due to the continuous entrainment of ambient air leads, under assumptions which are intuitively correct, to an expression analogous to aerodynamic drag. Thus the assumption that conventional aerodynamic drag acts on buoyantly-rising fireballs seems justified.

It is assumed that the speed of expansion of the rising fireball is small compared to the sound speed in the fireball. This will allow pressure equilibrium between the fireball and the ambient atmosphere. This assumption will be true for all possible cases of interest.

It is assumed that the rate of mass entrainment of ambient air can be described by the product of the surface area of the fireball, its speed relative to the atmosphere, the ambient density and a constant mass entrainment factor. It is further assumed that the entrained air is attached to the fireball at the ambient density. This implies that the entrained air is instantaneously accelerated to the speed of the fireball.

The analysis starts when the initial acceleration of the fireball, due to the "pressure force" and the buoyancy force, has given way to a quasi-stationary behavior of the rise speed. Also, fireball cooling by radiation must have become less important than cooling by entrainment or cooling by expansion in order for the analysis to be applicable. It is difficult to give a criterion for the time interval between the burst and the time when radiation cooling becomes unimportant. On the other hand, it is fairly easy to give the time interval after which a quasi-stationary rise speed is attained. For small fireballs, that is, for fireballs in which the rise speed due to the "pressure force" is small compared to that due to the buoyant force, the fireball will rise with significant speed once the above mentioned quasi-stationary behavior of rise speed has been attained. In general, entrainment and, consequently, cooling by entrainment may be expected to start once the fireball has acquired its quasi-stationary rise speed. The fireball acceleration time is therefore the minimum time after which the analysis will be applicable.

It is assumed that no heat is exchanged between the fireball and the entrained air. This assumption is unrealistic, since actually the entrained air removes heat from the fireball. This increases the density of the hot fireball gas and decreases the density of the entrained air. The assumption that no heat exchange between the hot fireball gas and the entrained air takes place is therfore similar to averaging between the truly increasing fireball density and truly decreasing density of the entrained air. An assessment of the validity of this procedure must await a more detailed study of the heat exchange process.*

It is furthermore assumed that both fireball gas and entrained air change their temperature adiabatically as they rise and expand. This is a perfectly tenable procedure, since adiabatic expansion actually takes place at times when energy loss by radiation has become unimportant. At late times, when the fireball has cooled sufficiently, the fireball gas starts to behave thermodynamically like ambient air and can be described by a constant ratio of specific heats.

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* See page 49.

It is assumed that the fireball radius and its average density and temperature are known at the time when the analysis starts to become applicable. A numerical value for the radius is much more easily guessed at than values for average density and temperature.

5. The Equation of Motion of the Buoyant Rise

It is convenient to start the analysis of buoyant rise by considering the equation of motion of the buoyantly accelerated fireball. The equation of motion is

$$d\left[M(1+\rho_A/2\overline{\rho}_F)\,v\right]/dt = (\rho_A-\overline{\rho}_F)\,Mg/\overline{\rho}_F - C_d\,\rho_A A v^2/2 \quad . \tag{5.1}$$

M stands for the fireball mass; ρ_A, $\overline{\rho}_F$ for the ambient, average fireball density; v for the velocity of the fireball; g for the gravitational acceleration; C_d for drag coefficient and A for cross-sectional area of the fireball.

It is easy to see the physical significance of the terms. An exception is the inclusion of the additive term $\rho_A/2\overline{\rho}_F$ into the first parenthesis on the left. This term takes the force into account which appears when the accelerated fireball displaces the ambient atmosphere (see Appendix 1). The expression for the drag is the conventional one for the motion of a solid sphere in a gas. The drag coefficient is a function of the Reynolds number and depends upon rise speed, ambient density and fireball radius (see Appendix 2). The entrainment of ambient air into the fireball is included by differentiating the momentum rather than the speed with respect to time. Carrying out the differentiations and multiplying by $2\overline{\rho}_F/M\rho_A$ leads, after rearrangement of terms, to

$$\left(1+\frac{2\overline{\rho}_F}{\rho_A}\right)\left(\frac{v}{M}\frac{dM}{dt}+\frac{dv}{dt}\right)+\frac{v}{\rho_A}\frac{d\rho_A}{dt}-\frac{v}{\overline{\rho}_F}\frac{d\overline{\rho}_F}{dt} = 2g\left(1-\frac{\overline{\rho}_F}{\rho_A}\right)-\frac{C_d\overline{\rho}_F A v^2}{M} \quad . \tag{5.2}$$

Any change in fireball volume V is due to two causes. One cause is the expansion of the rising fireball in equilibrium with the instantaneous ambient pressure. For the case that no ambient air is entrained into the fireball, its mass is conserved, and it follows that any change in its volume V due to expansion is given by

$$dV_{ex} = V\left(\frac{d\overline{T}_F}{\overline{T}_F}-\frac{dp_A}{p_A}\right) \quad . \tag{5.3}$$

This equation follows from the equation of state of the expanding fireball which is in pressure equilibrium with the ambient atmosphere

$$p_A V = M \overline{R}_F \overline{T}_F \tag{5.4}$$

by differentiation. The ambient pressure is denoted by p_A, the average fireball temperature is $\overline{T}_F$, the specific gas constant of the average fireball gas is $\overline{R}_F$. The quantities $\overline{T}_F$ and $\overline{R}_F$ are defined in Appendix 3. Use of the adiabatic equation of state for the average fireball gas

$$\overline{T}_F \, p_A^{\frac{1-\overline{\gamma}_F}{\overline{\gamma}_F}} = \text{const} \quad , \tag{5.5}$$

for substitution of $d\overline{T}_F/\overline{T}_F$ in Equation (5. 3) leads to

$$dV_{ex} = -\frac{V}{\overline{\gamma}_F} \frac{dp_A}{p_A} \quad . \tag{5.6}$$

The exponent $\overline{\gamma}_F$ in the adiabatic equation of state is defined in Appendix 4.

The change in ambient pressure can be described by the change in ambient density

$$d\rho_A/\rho_A \simeq -dh/H_{s\rho} \quad , \tag{5.7}$$

and the change in ambient temperature

$$dT_A/T_A \simeq -dh/H_{sT} \quad . \tag{5.8}$$

Density as well as temperature are described by scale heights. The scale height for the density distribution is $H_{s\rho}$. The scale height for the temperature distribution is H_{sT}. Their values are discussed in Appendix 5. With Equations (5. 7) and (5. 8) the ratio dp_A/p_A is

$$\frac{dp_A}{p_A} = -dh\left(\frac{1}{H_{s\rho}} + \frac{1}{H_{sT}}\right) \quad . \tag{5.9}$$

By introducing the equivalent scale height H_{se}

$$\frac{1}{H_{se}} = \frac{1}{H_{s\rho}} + \frac{1}{H_{sT}} \quad , \tag{5.10}$$

and substitution for dh from Equation (5. 7) the change in fireball volume (5. 6) due to expansion is

$$dV_{ex} = -\frac{V}{\gamma_F}\frac{H_{s\rho}}{H_{se}}\frac{d\rho_A}{\rho_A} \quad . \tag{5. 11}$$

The other cause for a change in fireball volume is the entrainment of ambient air. For a mass dM which is entrained into the fireball, the fireball volume will increase by

$$dV_e = dM/\rho_{e,i} \quad . \tag{5. 12}$$

The entrained air has the density $\rho_{e,i}$ after having entered the fireball. The density $\rho_{e,i}$ is the instantaneous ambient density in case the entrained air is rapidly accelerated to the fireball's rise speed. The total change in fireball volume is due to the sum of the two mentioned changes in volume

$$dV = \frac{dM}{\rho_{e,i}} - \frac{V}{\gamma_F}\frac{H_{s\rho}}{H_{se}}\frac{d\rho_A}{\rho_A} \quad . \tag{5. 13}$$

The change in the average fireball density is therefore

$$d\overline{\rho}_F = d\left(\frac{M}{V}\right) = \overline{\rho}_F\left(1 - \frac{\overline{\rho}_F}{\rho_A}\frac{\rho_A}{\rho_{e,i}}\right)\frac{dM}{M} + \frac{\overline{\rho}_F}{\gamma_F}\frac{H_{s\rho}}{H_{se}}\frac{d\rho_A}{\rho_A} \quad . \tag{5. 14}$$

(This equation is very important and will be used in a different context.)

Substitution of expression (5. 14) into the equation of motion for the buoyant rise (5. 2) and rearrangement results in

$$\left(1+\frac{2\overline{\rho}_F}{\rho_A}\right)\frac{dv}{dt} = 2g\left(1-\frac{\overline{\rho}_F}{\rho_A}\right)-\left(2+\frac{\rho_A}{\rho_{e,i}}\right)\frac{\overline{\rho}_F}{\rho_A}\frac{v}{M}\frac{dM}{dt} - \frac{v}{\rho_A}\left(1-\frac{1}{\gamma_F}\frac{H_{s\rho}}{H_{se}}\right)\frac{d\rho_A}{dt} - \frac{C_d\overline{\rho}_F A v^2}{M} \quad . \tag{5. 15}$$

The mass of air entrained per unit time dM/dt is assumed to be given by

$$\frac{dM}{dt} = \lambda\, 4\pi r_F^2\, \rho_A v \quad . \tag{5. 16}$$

The fireball radius is r_F, λ is the "mass entrainment factor." The expression is based on the intuitive feeling that a certain fraction of the total mass of ambient air with which the fireball comes into contact in rising a unit distance becomes permanently attached to the fireball. The "mass entrainment factor" λ may be determined by a comparison of theoretically computed with experimentally measured rise speeds.

Substitution of the expression (5.16) for the mass entrainment rate, of the expressions for the cross-sectional area and mass of a spherical fireball of radius r_F and average density $\overline{\rho}_F$, and the assumption of rapid acceleration of the entrained air to fireball rise speed, so that $\rho_{e,i} = \rho_A$, results in

$$\left(1+\frac{2\overline{\rho}_F}{\rho_A}\right)\frac{dv}{dt} = 2g\left(1-\frac{\overline{\rho}_F}{\rho_A}\right)-\left(9\frac{\lambda}{r_F}+\frac{3}{4}\frac{C_d}{r_F}\right)v^2+\frac{1}{H_{S\rho}}\left(1-\frac{1}{\overline{\gamma}_F}\frac{H_{S\rho}}{H_{Se}}\right)v^2 \quad . \qquad (5.17)$$

It is seen that entrainment produces a drag which is proportional to the square of velocity and inversely proportional to fireball radius just as the aerodynamic drag. It is also seen that in an isothermal atmosphere, that is in an atmosphere in which the scale height H_{ST} of temperature is very large, the fireball experiences a slight acceleration proportional to the square of its speed. This acceleration is due to the decreasing mass of air which the fireball displaces as it rises. This acceleration decreases from its value in an isothermal atmosphere for a positive value of H_{ST}, that is when the atmospheric temperature drops with height. Then the fireball expands more rapidly than in an isothermal atmosphere and displaces slightly more ambient air than in an isothermal atmosphere. The acceleration increases from its value in an isothermal atmosphere for negative values of H_{ST}, that is when the temperature increases with height. Then the fireball does not expand as rapidly as in an isothermal atmosphere and displaces slightly less air than in an isothermal atmosphere.

6. General Solution of the Equation of Motion of Buoyant Rise

In the differential equation of buoyant rise, only ρ_A is a known function of height. The latter is obtained as the first integral of rise speed. The rise speed, the fireball radius and mass and its average density are unknown. The average fireball density and the fireball mass can be computed from the differential Equations (5.14) and (5.16) Equation (5.14) contains the average $\overline{\gamma}_F$ of the fireball gases. In order to obtain it, the volumes of the various fireball components and their respective γ's have to be known, as it is shown in Appendix 4. Further equations are needed to complete the analysis.

Energy and mass balance equations for the hot fireball gas and for the entrained air have to be considered in connection with the condition of pressure equilibrium between hot fireball gas, entrained masses of air and the ambient atmosphere. In the energy balance equations, the heat lost by radiation and the particular way in which heat is exchanged between the hot fireball gas and the already attached and the newly entrained air must be taken into account. Temperatures and densities of the various fireball components as well as parameters describing the radiative properties of air as a function of temperature and density will appear in the mathematical expressions.

The fireball radius can be computed from a mass balance equation which, for the spherically shaped fireball considered here, is

$$\frac{4\pi}{3} r_F^3(t) = \frac{M_0}{\overline{\rho}_{F,i}(t)} + \int_0^t \frac{dM}{\overline{\rho}_e(t_e, t)} dt_e \quad . \tag{6.1}$$

$\overline{\rho}_{F,i}(t)$ refers to the spatial average of the hot fireball gas density at time t. $\overline{\rho}_e(t_e, t)$ refers to the spatial average at time t of the density of air entrained at time t_e. The initial mass of the hot fireball gas is M_0.

This concludes the discussion of the equations which have to be solved simultaneously to obtain the speed, mass and average density of a buoyantly rising fireball. The assumptions made so far concerned the particular form of the mass entrainment law, the assumption that the entrained air is attached to the fireball at its ambient density, the assumption of a spherical fireball shape and the use of the conventional aerodynamic drag relationship. The first mentioned assumption could be changed readily by using a different relationship for mass entrainment, but the latter three assumptions cannot be changed as easily. The problem is difficult to solve in the nearly complete form. It is possible to analytically study the behavior of most of the unknown quantities to a lesser degree of approximation by introducing further approximations.

DEVELOPMENT OF COMPUTATIONAL PROCEDURES

7. Quasi-Stationary Solution of the Equation of Buoyant Rise

A great simplification results when the equation of buoyant rise is considered in the quasi-stationary case. It arises when the buoyant force

$$2g\,(1 - \bar{\rho}_F/\rho_A) \tag{7.1}$$

is nearly cancelled by the sum of entrainment induced and aerodynamic drag

$$(9\lambda + 3/4\, C_d)\, v^2/r_F \quad , \tag{7.2}$$

and the accelerating or decelerating force

$$(1/H_{s\rho})\,(1 - H_{s\rho}/\bar{\gamma}_F H_{se})\, v^2 \quad . \tag{7.3}$$

A negligibly small fireball acceleration will be the result.

Under the assumption that the validity of the present investigation will be confined to that portion of the rise where quasi-stationary behavior prevails the rise speed can be expressed by

$$v = \left[\frac{2g\, r_F \left(1 - \dfrac{\bar{\rho}_F}{\rho_A}\right)}{9\lambda + 0.75\, C_d - \left(1 - \dfrac{1}{\gamma_F}\dfrac{H_{s\rho}}{H_{se}}\right)\dfrac{r_F}{H_{s\rho}}} \right]^{1/2} \quad . \tag{7.4}$$

It is seen that the speed will be a function of time mostly because of the variation with time of fireball radius and the ratio of average fireball density to ambient air density. The effect of the third term in the denominator is usually small compared to that of the first two terms, since buoyantly rising fireballs are always small compared to a scale height. This is particularly true for fireballs rising in an atmosphere in which the temperature decreases with altitude.

It is seen that for a small ratio of average fireball density to ambient air density, which prevails during the initial portion of the rise equation (7. 4) can be used to evaluate λ from experimentally measured values of rise speed and fireball radius. The drag coefficient must be evaluated for the proper Reynolds number. Pertinent data are given in Appendix 2. An approximate expression for the rise speed is obtained from experimental evidence as

$$v \approx 2 [g\, r_F]^{1/2} \quad . \tag{7. 5}$$

It is also seen that immediately after the burst when the average fireball density equals the ambient air density no buoyant acceleration exists. At the end of the expansion phase the average fireball density has dropped appreciably below the ambient air density and the buoyant acceleration approaches an upper limit of 2g. By neglecting drag while the fireball accelerates to its quasi-stationary speed, an initial acceleration time

$$t_{acc} \geq (r_F/g)^{1/2} \tag{7. 6}$$

is obtained.

The entrainment of ambient air is proportional to rise speed and will therefore be most significant after this initial period of acceleration.

8. The General Computation of the Average Fireball Density

The calculation of the average fireball density is greatly simplified when energy loss due to radiation can be neglected. The establishment of a relevant criterion is somewhat arbitrary. Since the present analysis is concerned with average values, the following criterion is proposed: Radiative energy loss is neglected when radiation decreases the average fireball temperature less rapidly than mass entrainment. Radiative cooling will certainly be important until mass entrainment becomes significant at the end of the initial acceleration phase, but even after this time it may control the radiative fireball temperature. It is difficult to give general numerical estimates for the duration of radiative cooling as it has been defined above, taking expansion and mass entrainment into account. In the following, the differential equation for the ratio of average fireball density to ambient air density will be developed, which is applicable once radiative energy loss can be neglected.

It follows from Equation (5.14) that for $\rho_{e,i} = \rho_A$

$$\frac{d(\bar{\rho}_F/\rho_A)}{\bar{\rho}_F/\rho_A} = \left(1 - \frac{\bar{\rho}_F}{\rho_A}\right)\frac{dM}{M} - \left(1 - \frac{1}{\bar{\gamma}_F}\frac{H_{s\rho}}{H_{se}}\right)\frac{d\rho_A}{\rho_A} \quad . \tag{8.1}$$

With Equation (5.7) for $d\rho_A/\rho_A$ and Equation (5.16) for dM/M, Equation (8.1) appears as

$$\frac{d(\bar{\rho}_F/\rho_A)}{\bar{\rho}_F/\rho_A} = \left(1 - \frac{\bar{\rho}_F}{\rho_A}\right)3\lambda\frac{\rho_A}{\bar{\rho}_F}\frac{dh}{r_F} + \left(1 - \frac{1}{\bar{\gamma}_F}\frac{H_{s\rho}}{H_{se}}\right)\frac{dh}{H_{s\rho}} \quad . \tag{8.2}$$

In this equation the first term on the right-hand side expresses the change in average fireball density due to mass entrainment, the second term the change due to adiabatic expansion.

As long as

$$3\lambda\frac{H_{s\rho}}{r_F}\left(\frac{\rho_A}{\bar{\rho}_F} - 1\right) > \left(1 - \frac{1}{\bar{\gamma}_F}\frac{H_{s\rho}}{H_{se}}\right) \quad , \tag{8.3}$$

mass entrainment determines the average fireball density. The phase of the rise where this is true will be referred to as the mass entrainment dominated phase. It is also seen that the average $\bar{\gamma}_F$ does not, under these circumstances appear in the differential equation (8.2). This equation can therefore be integrated in closed form if one assumes that the fireball radius remains constant over the height of integration. These simplifications will now be applied.

9. The Computation of the Average Fireball Density where Mass Entrainment Dominates

The differential equation for the average fireball density (8.2) containing the term due to mass entrainment only is

$$\frac{d(\bar{\rho}_F/\rho_A)}{1 - \bar{\rho}_F/\rho_A} = \frac{3\lambda}{r_F}\,dh \quad . \tag{9.1}$$

This differential equation can easily be integrated if one assumes r_F to be a constant denoted by $r_{F,k}$, while integrating from $h = 0$ to $h = r_{F,k}$. This is well justified because it will be seen later that $r_{F,k}$ varies only slowly with height. By integrating N times from $h = 0$ to $h = r_{F,k}$, keeping $r_{F,k}$ constant during every step but allowing it to vary from one step to the next

$$(\overline{\rho}_F/\rho_A)_N = 1 - [1 - (\overline{\rho}_F/\rho_A)_0]\, e^{-3\lambda N} \tag{9.2}$$

is obtained. The initial value, that is the value of $\overline{\rho}_F/\rho_A$ at the end of the radiative expansion phase, when the fireball has already accelerated to its quasi-stationary speed, is $(\overline{\rho}_F/\rho_A)_0$.

Convenient approximations for Equation (9. 2) are:

$$\text{for} \quad 3\lambda N \ll 1, \quad (\overline{\rho}_F/\rho_A)_N = (\overline{\rho}_F/\rho_A)_0 + 3\lambda N \quad , \tag{9.3}$$

$$\text{for} \quad (\overline{\rho}_F/\rho_A)_0 \ll 1, \quad (\overline{\rho}_F/\rho_A)_N = 1 - e^{-3\lambda N} \quad , \tag{9.4}$$

$$\text{and for} \quad 3\lambda N \ll 1 \text{ and } (\overline{\rho}_F/\rho_A)_0 \ll 1, \quad (\overline{\rho}_F/\rho_A)_N = 3\lambda N \quad . \tag{9.5}$$

The total height $h_{tot,N}$ over which integration has been carried out is

$$h_{tot,N} = \sum_{k=1}^{N} r_{F,k} \quad . \tag{9.6}$$

The value of $r_{F,k}$ will be computed in a following section.

The time $t_{tot,N}$ for the fireball to rise to the height $h_{tot,N}$ is the sum of the time intervals to rise the individual step heights $r_{F,k}$

$$t_{tot,N} = \sum_{k=1}^{N} \frac{r_{F,k}}{\left[\dfrac{2g\, r_{F,k}(1 - (\rho_F/\rho_A)_k)}{9\lambda + 0.75\, C_d - (1 - H_{S0}/\overline{\gamma}_{F,k} H_{Se})\, r_{F,k}/H_{S0}}\right]^{1/2}} \quad . \tag{9.7}$$

The value of the unknown quantity $\overline{\gamma}_F$ for the k^{th} step appears in this equation in a term of minor importance only. It is therefore adequate to use a very approximate value for $\overline{\gamma}_{F,k}$.

10. The Computation of the Fireball Mass when Mass Entrainment Dominates

The differential equation for mass entrainment (5. 16) can be integrated in analogy to Equation (9. 1) by integrating N times from h = 0 to $h = r_{F,k}$, keeping $r_{F,k}$ constant during every step. In this way, the equation

$$\frac{dM}{M} = 3\lambda \left(\frac{\rho_A}{\bar{\rho}_F}\right)_k \frac{dh}{r_{F,k}} \qquad (10.1)$$

goes by substitution with Equation (9.2) over into

$$\ln M = \int_0^N \frac{3\lambda}{1-[1-(\bar{\rho}_F/\rho_A)_0]e^{-3\lambda N}}\, dK + \text{constant} \quad . \qquad (10.2)$$

Integration of Equation (10.2) and use of M_0 for M leads to

$$\left(\frac{M}{M_0}\right)_N = \frac{1-[1-(\bar{\rho}_F/\rho_A)_0]\, e^{-3\lambda N}}{(\bar{\rho}_F/\rho_A)_0}\, e^{3\lambda N} \quad ,$$

$$= \frac{(\bar{\rho}_F/\rho_A)_N}{(\bar{\rho}_F/\rho_A)_0}\, e^{3\lambda N} \quad . \qquad (10.3)$$

Convenient approximations for Equation (10.3) are:

for

$3\lambda N \ll 1$, $$\left(\frac{M}{M_0}\right)_N = \frac{(\bar{\rho}_F/\rho_A)_0 + 3\lambda N}{(\bar{\rho}_F/\rho_A)_0}(1+3\lambda N) \ , \qquad (10.4)$$

for

$(\bar{\rho}_F/\rho_A)_0 \ll 1$, $$\left(\frac{M}{M_0}\right)_N = e^{3\lambda N} - 1 \quad , \qquad (10.5)$$

and for

$3\lambda N \ll 1$ and $(\rho_F/\rho_A)_0 \ll 1$, $$\left(\frac{M}{M_0}\right)_N = 1 + \frac{3\lambda N}{(\bar{\rho}_F/\rho_A)_0} \quad . \qquad (10.6)$$

11. The Computation of the Fireball Radius when Mass Entrainment Dominates

The ratio of fireball mass to original mass after having risen N instantaneous radii is

$$\left(\frac{M}{M_0}\right)_N = \frac{\bar{\rho}_{F,N}\, r_{F,N}^3}{\bar{\rho}_{F,0}\, r_{F,0}^3} \quad . \qquad (11.1)$$

From this one obtains by simple transformation

$$r_{F,N} = \left[\left(\frac{M}{M_0}\right)_N \left(\frac{\bar{\rho}_F}{\rho_A}\right)_0 \frac{1}{(\bar{\rho}_F/\rho_A)_N} \left(\frac{\rho_{A,0}}{\rho_{A,N}}\right) \right]^{1/3} r_{F,0} \quad . \tag{11.2}$$

The substitution of Equations (9. 2) and (10. 3), and the assumption of an atmospheric density declining with scale height $H_{s\rho}$ leads to

$$r_{F,N} = r_{F,0}\, e^{\left(\lambda_N + \frac{h_{tot,N}}{3H_{s\rho}}\right)} \quad . \tag{11.3}$$

By neglecting the growth of the fireball radius,

$$N \leq h_{tot,N}/r_{F,0} \simeq v\, t_{tot,N}$$

$$r_{F,N} \leq r_{F,0}\, e^{\left(\lambda + \frac{r_{F,0}}{3H_{s\rho}}\right) \frac{v\, t_{tot,N}}{r_{F,0}}} \quad . \tag{11.4}$$

An estimate of the expansion velocity $\dot{r}_{F,N}$ is obtained by differentiation of Equation (11. 3)

$$\dot{r}_{F,N} < \left(\lambda + \frac{r_{F,0}}{3H_{s\rho}} \cdot e^{\left(\lambda + \frac{r_{F,0}}{3H_{s\rho}}\right) \frac{v\, t_{tot,N}}{r_{F,0}}} \right) v \quad . \tag{11.5}$$

It is seen that the speed of expansion is smaller than, or at most about equal to the rise speed v, provided the initial fireball radius is small compared to a scale height and provided the fireball has risen no more than several scale heights. It can be shown that fireballs exceeding a certain size are increasingly more influenced by the "pressure force" mentioned before, than by buoyancy forces (see Reference 1). The fireball radius when the transition starts to take place is difficult to determine but is certainly larger than about one-fifth of a scale height. It's most likely value is five-to seven-tenths of a scale height. Numerical examples show that the entrainment controlled phase of nuclear-fireball rise rarely exceeds a few scale heights. For these reasons the assumption of buoyant fireball rise is justified for fireballs of less radius than five-tenths of a scale height.

In case the scale height changes with altitude, the fireball radius should be computed from

$$r_{F,N} = r_{F,N-1}\, e^{\left(\lambda + \frac{\Delta h_N}{3H_{s\rho}}\right)} \quad , \tag{11.6}$$

by using the instantaneous value of the scale height $H_{s\rho}$.

12. The Computation of the Average Fireball Temperature when Mass Entrainment Dominates*

The condition of pressure equilibrium between the fireball gas and the ambient atmosphere allows to compute the average fireball temperature after the fireball has risen N instantaneous radii as

$$\overline{T}_{F,N} = \frac{R_A}{\overline{R}_{F,N}} \, \frac{\rho_{A,N}}{\overline{\rho}_{F,N}} \, T_{A,N} \quad . \tag{12.1}$$

The specific average gas constant of the fireball $\overline{R}_{F,N}$ can be obtained from Equation (A3. 7) of Appendix 3.

The average temperature of the fireball is defined in Appendix 3.

13. Graphs of $(\overline{\rho}_F/\rho_A)_N$, $(M/M_0)_N$ and $1/(\overline{\rho}_F/\rho_A)_N$

It is seen that the expressions for $(\overline{\rho}_F/\rho_A)_N$ and $(M/M_0)_N$ depend only upon the product of $\lambda \cdot N$ and the initial conditions. The mentioned functions can therefore easily be plotted. This is done in Figures 13. 1 through 13. 3. Figure 13. 4 gives the factor $1/(\overline{\rho}_F/\rho_A)_N$ which is needed for a determination of the average fireball temperature. It is good to keep in mind that the abscissa in all these figures is the product of $\lambda \cdot N$, so that a different mass entrainment factor λ implies a different N. Since N is the number of instantaneous radii the fireball has risen, it is also a measure for the fireball height $h_{tot,e}$ above the burst height, to which the fireball has ascended in entrainment dominated rise. By neglecting the growth of the fireball radius

$$h_{tot,N} \gtrsim N r_{F,0} \quad . \tag{13.1}$$

From

$$\begin{aligned} h_{tot,N} &\simeq v\, t_{tot,N} \quad , \\ &\simeq 2[g\, r_{F,0}]^{1/2}\, t_{tot,N} \end{aligned} \tag{13.2}$$

and Equation (10. 4) it follows that

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* See Page 49.

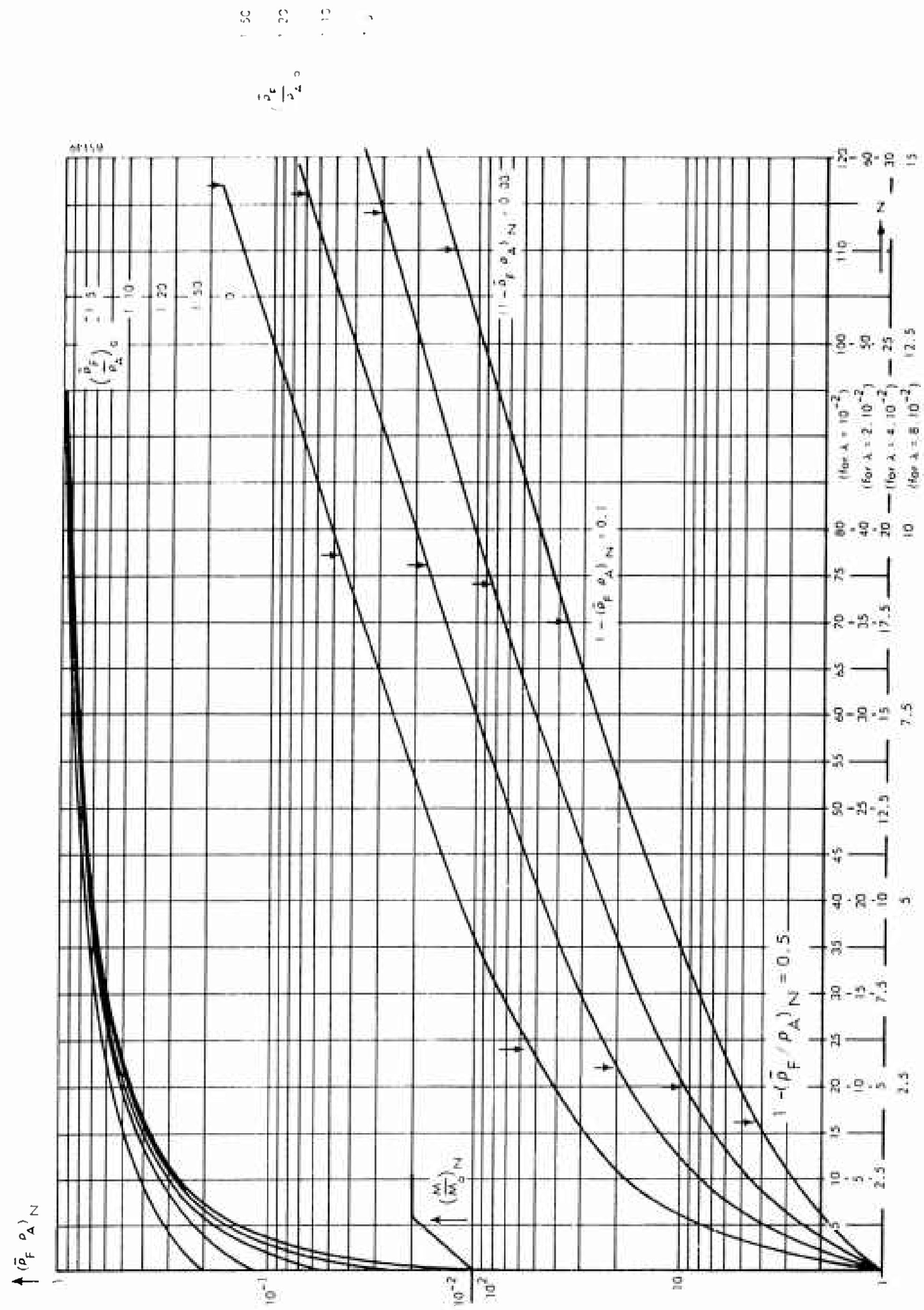

Figure 13.1. Density and Mass Ratios for Entrainment Dominated Fireballs

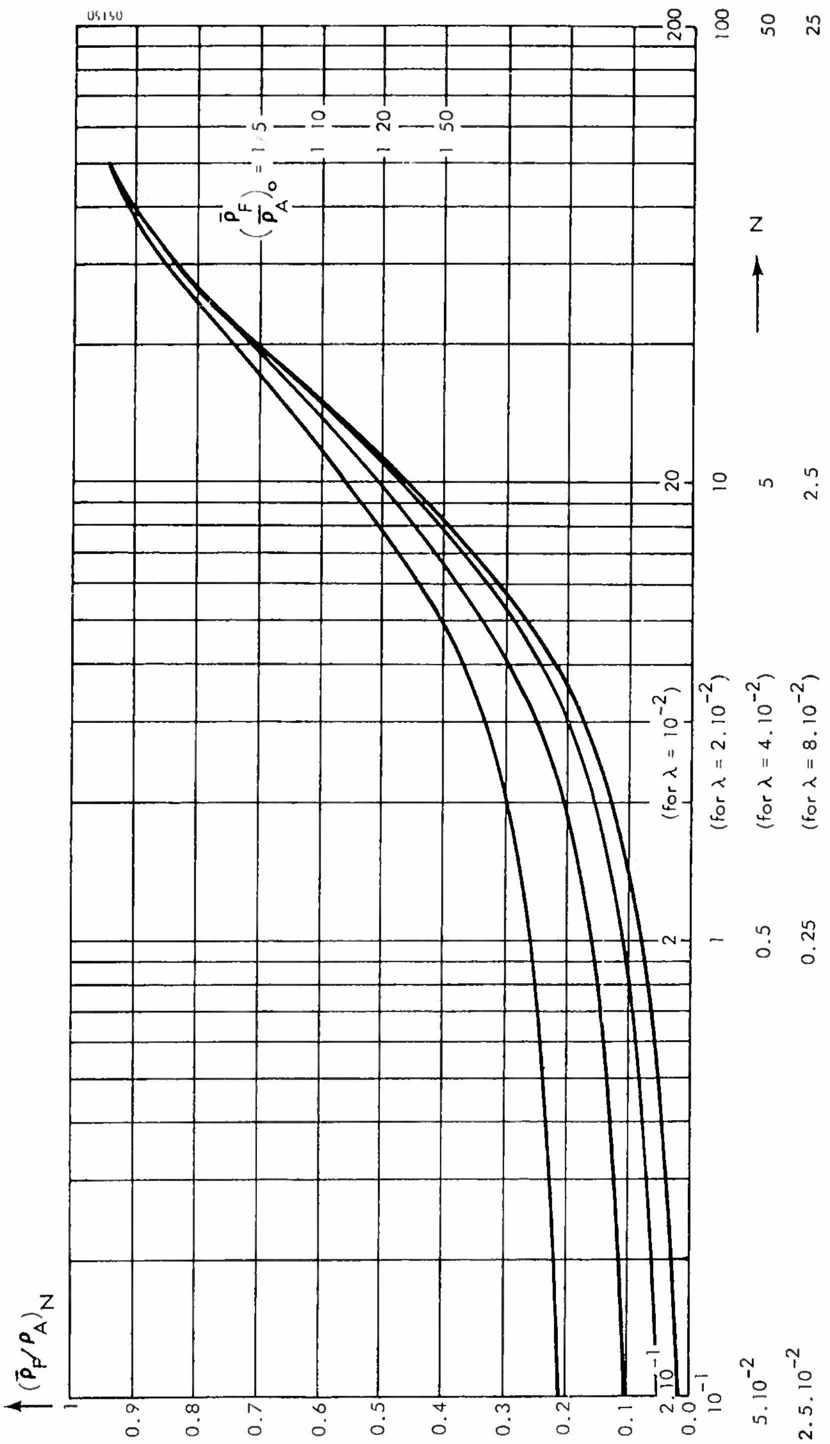

Figure 13. 2. Density Ratios for Entrainment Dominated Fireballs

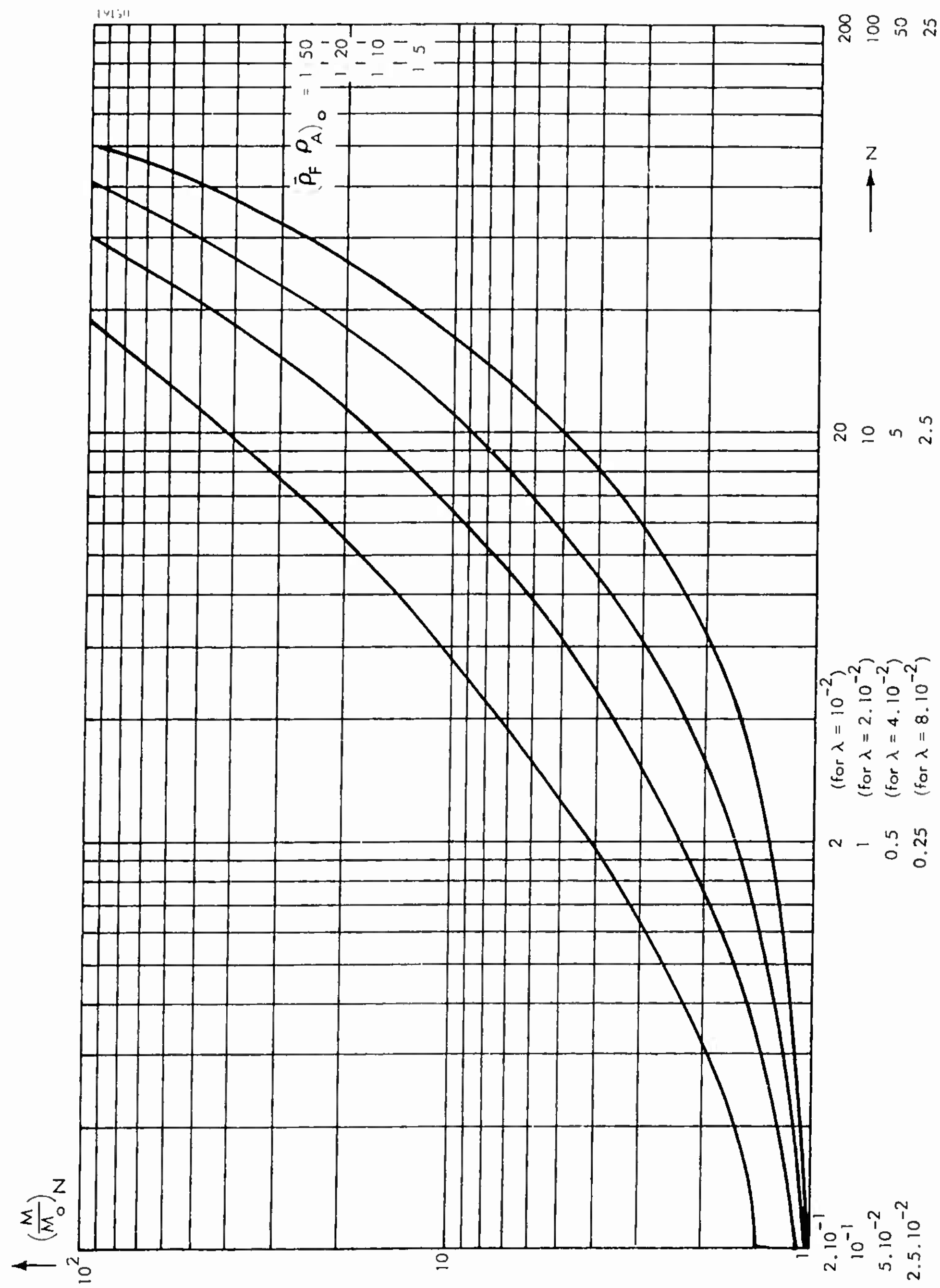

Figure 13. 3. Mass Ratios for Entrainment Dominated Fireballs

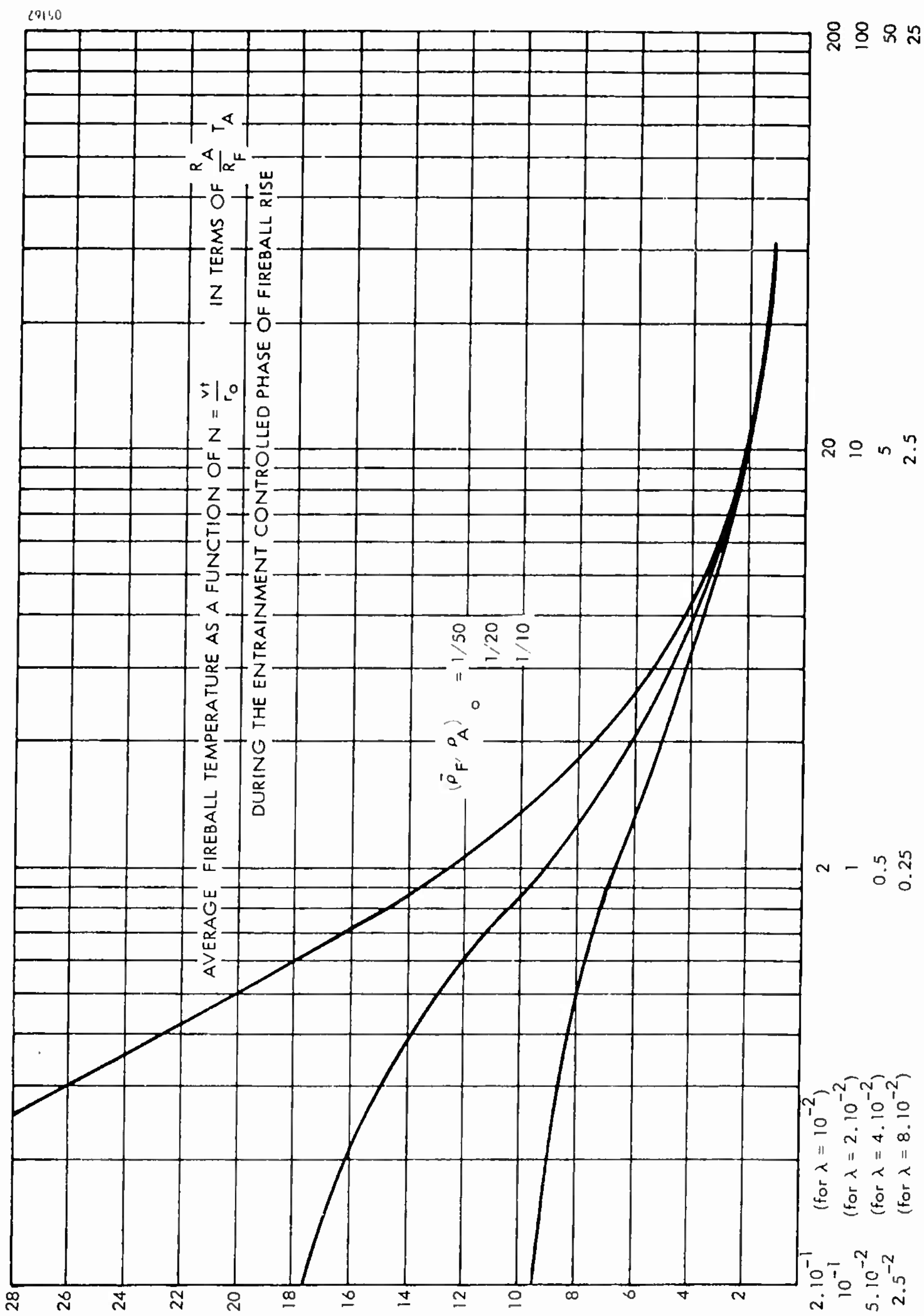

Figure 13.4. Temperature Factors for Entrainment Dominated Fireballs

$$t_{tot,N} \geq N r_{F,0}/v \quad ,$$
$$\simeq N r_{F,0}^{1/2}/2g^{1/2} \quad . \qquad (13.3)$$

14. The Computation of the Average Fireball Density when Adiabatic Expansion Dominates

The differential equation (8. 2) for the average fireball density containing only the term due to adiabatic expansion is

$$\frac{d(\overline{\rho}_F/\rho_A)}{\overline{\rho}_F/\rho_A} = \left(1 - \frac{1}{\overline{\gamma}_F}\frac{H_{S\rho}}{H_{Se}}\right)\frac{dh}{H_{S\rho}} \quad . \qquad (14.1)$$

This differential equation describes the average fireball density once inequality (8. 3) reverses. This phase of the rise during which the average fireball density is determined by adiabatic expansion rather than entrainment, will be referred to as the expansion-dominated phase of the rise. The value of $\overline{\gamma}_F$ will now be close to that of ambient air, since the fireball has been cooled by entrainment.

Upon integration the ratio of average fireball density to ambient air density at height h above that height where entrainment ceased to dominate is found to be

$$(\overline{\rho}_F/\rho_A)_h = (\overline{\rho}_F/\rho_A)_N \, e^{\left(1 - \frac{1}{\overline{\gamma}_F}\frac{H_{S\rho}}{H_{Se}}\right)\frac{h}{H_{S\rho}}} \quad . \qquad (14.2)$$

The ratio $(\overline{\rho}_F/\rho_A)_N$ refers to the ratio at the termination of the entrainment-controlled phase of the rise.

The height $h_{tot,a}$ for which the fireball will rise controlled by adiabatic expansion until its average density is equal to the ambient air density so that the buoyancy force disappears is

$$h_{tot,a} = \frac{\ln \frac{1}{(\overline{\rho}_F/\rho_A)_N}}{1 - \frac{1}{\overline{\gamma}_F}\frac{H_{S\rho}}{H_{Se}}} H_{S\rho} \quad . \qquad (14.3)$$

The ratio $(\overline{\rho}_F/\rho_A)_N$ is the ratio of $\overline{\rho}_F/\rho_A$ at the end of the entrainment-dominated phase of the fireball rise. Its value is obtained from expression (9. 2) for that N for which $r_{F,N}$ and $(\overline{\rho}_F/\rho_A)_N$ assume such magnitudes that the inequality (8. 3) ceases to be valid.

In case the scale heights change with altitude the ratio of average fireball density to ambient air density should be computed step by step from

$$(\overline{\rho}_F/\rho_A)_h = (\overline{\rho}_F/\rho_A)_{h-\Delta h}\, e^{\left(1 - \frac{1}{\overline{\gamma}_F}\frac{H_{s\rho}}{H_{se}}\right)\frac{\Delta h}{H_{s\rho}}} \qquad (14.4)$$

by using the instantaneous scale heights $H_{s\rho}$ and H_{se}.

15. The Computation of the Fireball Mass when Adiabatic Expansion Dominates

The computation of the average fireball density during that portion of its rise when adiabatic expansion dominates the average fireball density is based upon the neglect of mass entrainment. As a consequence the fireball mass remains constant. Its value is obtained from expression (10. 3) for that N for which $r_{F,N}$ and $(\overline{\rho}_F/\rho_A)_N$ assume such magnitudes that the inequality (8. 3) ceases to be valid.

16. The Computation of the Fireball Radius when Adiabatic Expansion Dominates

The fireball mass is constant during the expansion controlled portion of the fireball rise. For this reason it follows that

$$r^3_{F,h}\, \overline{\rho}_{F,h} = r^3_{F,N}\, \overline{\rho}_{F,N} \quad . \qquad (16.1)$$

By simple transformation

$$r_{F,h} = r_{F,N}\left(\frac{\overline{\rho}_F}{\rho_A}\right)^{1/3}_N \left(\frac{\rho_A}{\overline{\rho}_F}\right)^{1/3}_h \left(\frac{\rho_{A,N}}{\rho_{A,h}}\right)^{1/3} \quad . \qquad (16.2)$$

With the help of Equations (5. 7) and (14. 2), Equation (16. 2) becomes

$$r_{F,h} = r_{F,N}\, e^{\frac{1}{3\overline{\gamma}_F}\frac{h}{H_{se}}} \quad . \qquad (16.3)$$

In this equation $r_{F,h}$ is the fireball radius at height h above that point where adiabatic expansion started to dominate, $r_{F,N}$ is the fireball radius where transition from entrainment limited to expansion-controlled rise took place.

An estimate of the expansion velocity is obtained by differentiation of Equation (16. 3) under the approximation that $h \simeq v\, t_{tot}$.

$$\dot{r}_{F,h} = \frac{1}{3\bar{\gamma}_F} \frac{r_{F,N}}{H_{Se}}\, e^{\frac{1}{3\bar{\gamma}_F}\frac{h}{H_{Se}}}\, v \quad . \tag{16.4}$$

It is seen that the speed of expansion is small compared to the rise speed v, provided the fireball radius at the beginning of the expansion-controlled phase is small compared to the scale height H_{Se}, and provided the fireball has risen no more than several scale heights H_{Se}. This result is quite similar to that obtained for the expansion speed during the entrainment-controlled portion of the rise. Therefore the same reasoning used there also applies here. As a result it can be concluded that all fireballs of less radius than about one-third of a scale height rise buoyantly, that is in pressure equilibirum with the ambient atmosphere.

In case the scale heights change with altitude the fireball radius should be computed step by step from

$$r_{F,h} = r_{F,h-\Delta h}\, e^{\frac{1}{3\bar{\gamma}_F}\frac{\Delta h}{H_{Se}}} \tag{16.5}$$

by using the instantaneous scale height H_{Se} .

17. The Computation of the Average Fireball Temperature when Adiabatic Expansion Dominates*

The computation of the average fireball temperature when adiabatic expansion dominates proceeds exactly as it does in case entrainment dominates.

18. Computational Example of Entrainment and Expansion-Controlled Buoyant Fireball Rise

The following table gives an example for the practical numerical evaluation of buoyant fireball rise from formulae derived in the present report.

Initial values of fireball radius $r_{F,0}$ and of the ratio of average fireball density to ambient air density $(\bar{\rho}_F/\rho_A)_0$ are assumed. The burst height is taken as 20 km. The mass entrainment parameter λ is

* See Page 49.

assumed to be $5 \cdot 10^{-2}$. The fireball radius during the entrainment-dominated portion of the rise is computed from Equation (11. 3), the ratio of average fireball density to ambient air density from Equation (9. 2), the rise speed from Equation (7. 4). The height to which the fireball has risen and the time for the rise are computed from Equations (9. 6) and (9. 7).

The transition from entrainment dominated to expansion-controlled rise is determined from inequality(8. 3). During the expansion-controlled phase of the rise the fireball radius follows from Equation (16. 3), the ratio of average fireball density to ambient density from Equation (14. 2), the rise speed from Equation (7. 4). The equivalent scale height is quoted in Equation (5. 10). The various scale heights of interest are listed in Appendix 5. It is also assumed that the scale heights do not change with altitude. In case it is desired to take changing scale heights into account, Equation (11. 6) should be used instead of Equation (11. 3) for the computation of the fireball radii during entrainment-dominated rise, and Equation (16. 5) instead of Equation (16. 3) during expansion-dominated rise; Equation (14. 4) instead of Equation (14. 2) should be used to compute the fireball density during the expansion-dominated phase.

N	$r_{F,N} = r_{F,0}\, e^{\lambda N + \frac{h_{tot,N-1} + r_{N-1}}{3H_{sc}}}$, $H_{sc} = 6.4$ km, $\lambda = 5 \cdot 10^{-2}$	$h_{tot,N} = \sum_{k=1}^{N} r_{F,k}$	$(\bar{\rho}_F/\rho_A)_N = 1 - \left[1 - \left(\frac{\bar{\rho}_F}{\rho_A}\right)_0\right] e^{-3\lambda N}$	$V_N = \left[\frac{2g\, r_{F,N}\left(1 - \left(\frac{\bar{\rho}_F}{\rho_A}\right)_N\right)}{9\lambda + 0.75\, C_d - \left(1 - \frac{1}{\bar{\gamma}_{F,N}} \frac{H_{s\rho}}{H_{s\theta}}\right) \frac{r_{F}}{H_{s}}} \right.$
[0]	[km]	[km]	[0]	[km sec^{-1}]
0	$\underline{r_{F,0} = 0.83}$	0	$\underline{0.05}$	0.187 0.187*
1	$r_{F,1} = 0.83 \cdot e^{(5 \cdot 10^{-2} + 4.3 \cdot 10^{-2})} = 0.91$	0.91	0.185	0.182 0.182*
2	$r_{F,2} =$ " $\cdot\, e^{(10^{-1} + 9.5 \cdot 10^{-2})} = 1.01$	1.92	0.30	0.179 0.177*
3	$r_{F,3} =$ " $\cdot\, e^{(1.5 \cdot 10^{-1} + 1.52 \cdot 10^{-1})} = 1.13$	3.05	0.40	0.176 0.174*
4	$r_{F,4} =$ " $\cdot\, e^{(2 \cdot 10^{-1} + 2.17 \cdot 10^{-1})} = 1.26$	4.31	0.48	0.174 0.171*
5	$r_{F,5} =$ " $\cdot\, e^{(2.5 \cdot 10^{-1} + 2.9 \cdot 10^{-1})} = 1.42$	5.73	0.55	0.173 0.169*
6	$r_{F,6} =$ " $\cdot\, e^{(3 \cdot 10^{-1} + 3.7 \cdot 10^{-1})} = 1.63$	7.36	0.615	0.174 0.167*

* The asterisk indicates values obtained using the approximation expression $V_N \approx \left[\frac{2g r_{F,N}(1 - \bar{\rho}_F/\rho_A)_N}{9\lambda}\right]^{1/2}$. When the drag coefficient C
calculation where it is 0.05, the aforementioned approximation is justified since $\left(1 - \frac{1}{\bar{\gamma}_{F,N}} \frac{H_{s\rho}}{H_{s\theta}}\right) \frac{r_{F,N}}{H_{s\rho}} \sim 0.75\, C_d$. When not otherwise in
expression has been used to evaluate V_N.

h	$r_{F,h} = r_{F,N}\, e^{\frac{1}{3\bar{\gamma}_F} \frac{h}{H_{s\theta}}} = 1.63\, e^{0.033h}$ $H_{s\theta} = 7.3$ km, $H_{s\rho} = 6.6$ km, $\bar{\gamma}_F = 1.4$	$h_{tot,a} =$ 7.36 + h	$(\bar{\rho}_F/\rho_A)_h = (\bar{\rho}_F/\rho_A)_N\, e^{\frac{1}{\bar{\gamma}_F} \frac{h}{H_{s\theta}}}$	$V_h = \left[\frac{2g r_{F,h}(1 - (\bar{\rho}_F/\rho_A)_h)}{9\lambda + 0.75\, C_d - \left(1 - \frac{1}{\bar{\gamma}_{F,N}} \frac{H_{s\rho}}{H_{s\theta}}\right) \frac{r_{F,h}}{H_{s\rho}}} \right]^{1/2\,\dagger}$	$\Delta t_h = \frac{\Delta h}{V_h}$
[km]	[km]	[km]	[0]	[km sec^{-1}]	[sec]
3.0	$r_{F,3.0} = 1.63\, e^{0.1} = 1.63 \cdot 1.105 = 1.80$	10.36	$0.615\, e^{0.29} = 0.83$	0.038	$\gtrsim 7.9$
4.95	$r_{F,4.95} =$ " $e^{0.16} = 1.63 \cdot 1.177 = 1.92$	12.31	$0.615\, e^{0.485} = 1.00$	0	

† The rise speed has been computed under the assumption that the drag coefficients C_d remains constant at about 0.1. This is appr
of the order of 10^9.

Table 18.1. Computation of Entrainment and Expansion Control Buoyant Rise.

$= 6.4$ km, $\cdot = 5.10^{-2}$	$h_{tot,N} = \sum_{k=1}^{N} r_{F,k}$ [km]	$(\bar{\rho}_F/\rho_A)_N = 1 - \left[1 - \left(\frac{\bar{\rho}_F}{\rho_A}\right)_0\right] e^{-3\lambda N}$ [0]	$V_N = \left[\dfrac{2g\, r_{F,N}\left(1 - \left(\frac{\bar{\rho}_F}{\rho_A}\right)_N\right)}{9\lambda + 0.75\, C_d - \left(1 - \frac{1}{\bar{\gamma}_{F,N}}\frac{H_{S\rho}}{H_{S\theta}}\right)\frac{r_{F,N}}{H_{S\rho}}}\right]^{1/2\,*}$ [km sec^{-1}]	$\Delta t_N = \dfrac{r_{F,N}}{V_N}$ [sec]	$t_{tot,N} = \sum_{k=1}^{N} \Delta t_k$ [sec]	$(M/M_0)_N = \dfrac{(\bar{\rho}_F/\rho_A)_N}{(\bar{\rho}_F/\rho_A)_0} e^{3\lambda N}$ [0]
	0	0.05	0.187 0.187*	4.4		1.0
$\cdot 10^{-2}$) = 0.91	0.91	0.185	0.182 0.182*	5.0	5.0	4.3
0^{-2}) = 1.01	1.92	0.30	0.179 0.177*	5.7	10.7	8.1
.52 $\cdot 10^{-1}$) = 1.13	3.05	0.40	0.176 0.174*	6.4 6.5*	17.1 17.2*	12.5
7 $\cdot 10^{-1}$) = 1.26	4.31	0.48	0.174 0.171*	7.3 7.4*	24.4 24.6*	17.5
.9 $\cdot 10^{-1}$) = 1.42	5.73	0.55	0.173 0.169*	8.2 8.4*	32.6 33.0*	23.3
$\cdot 10^{-1}$) = 1.63	7.36	0.615	0.174 0.167*	9.4 9.8*	42.0 42.8*	30.3

d using the approximation expression $V_N \approx \left[\dfrac{2g r_{F,N}(1 - \bar{\rho}_F/\rho_A)_N}{9\lambda}\right]^{1/2}$. When the drag coefficient C_d is sufficiently small, as it is in this

entioned approximation is justified since $\left(1 - \frac{1}{\bar{\gamma}_{F,N}}\frac{H_{S\rho}}{H_{S\theta}}\right)\frac{r_{F,N}}{H_{S\rho}} \sim 0.75\, C_d$. When not otherwise indicated by the asterisk, the complete

.

m, $\bar{\gamma}_F = 1.4$	$h_{tot,a} = 7.36 + h$ [km]	$(\bar{\rho}_F/\rho_A)_h = (\bar{\rho}_F/\rho_A)_N\, e^{\frac{1}{\bar{\gamma}_F}\frac{h}{H_{S\theta}}}$ [0]	$V_h = \left[\dfrac{2g r_{F,h}(1 - (\bar{\rho}_F/\rho_A)_h)}{9\lambda + 0.75\, C_d - \left(1 - \frac{1}{\bar{\gamma}_{F,N}}\frac{H_{S\rho}}{H_{S\theta}}\right)\frac{r_{F,h}}{H_{S\rho}}}\right]^{1/2\,\dagger}$ [km sec^{-1}]	$\Delta t_h = \dfrac{\Delta h}{V_h}$ [sec]	$t_{tot} = t_{tot,N} + \sum_h \Delta t_h$ [sec]	$(M/M_0)_h = (M/M_0)_N$ [0]
= 1.80	10.36	$0.615\, e^{0.29} = 0.83$	0.038	$\gtrsim 7.9$	50.4 ± 0.4	30.3
= 1.92	12.31	$0.615\, e^{0.485} = 1.00$	0			30.3

ler the assumption that the drag coefficients C_d remains constant at about 0.1. This is approximately true for Reynolds numbers

ainment and Expansion Control Buoyant Rise.

APPENDIX 1

ON THE EQUATION OF MOTION OF THE BUOYANT RISE

The addition of the term $M\rho_A/2\overline{\rho}_F$ to the mass of the fireball in the equation of motion takes the pressure forces into account which appear during acceleration of a solid sphere in an infinite mass of incompressible fluid which is at rest at infinity. The expression has been taken from H. Lamb's book on Hydrodynamics, Dover, Inc., N. Y., 6th edition, p. 124.

The mentioned condition corresponds to the model used in the present report where the fireball is assumed to be of spherical shape. In addition the rise speed is almost always subsonic and thus the assumption that the ambient atmosphere behaves like an incompressible fluid is valid. The term $M\rho_A/2\overline{\rho}_F$ is just one-half of the ambient air displaced by the fireball. It represents the "equivalent mass" of air which must flow around the fireball to let it pass. Its contribution limits the acceleration which fireballs of very low average density and therefore very small mass can receive.

Once the acceleration of the fireball stops and a quasi-stationary behavior of the fireball speed is established, aerodynamic and entrainment induced drag and drag due to the change in fireball radius and thus in the mass of displaced ambient air control the fireball's speed.

APPENDIX 2

SUBSONIC DRAG OF A SOLID SPHERE

In Figure A2. 1 the subsonic drag coefficient of a solid sphere is plotted as a function of the Reynolds number. The drag coefficient C_d allows to compute the drag force F_d acting upon a sphere of cross-sectional area A moving at speed v through a medium of density ρ_A

$$F_d = C_d \rho_A A v^2/2 \quad . \tag{A2. 1}$$

The force is due to pressure differentials in the gas surrounding the sphere and not to frictional forces between the sphere's surface and the ambient gas. The pressure differentials are produced by boundary layer separation and by the generation of waves. For some of the range of Reynolds numbers of interest in fireball rise, the surface frictional drag is small compared to pressure drag. For Reynolds numbers larger than 10^7 to 10^8 no values for the drag coefficient could be found. It may be reasonable to assume that the drag coefficient decreases slowly above a Reynold's number of 10^8 and has a value somewhere between 10^{-2} and 10^{-1}.

The Reynolds number is defined as

$$R_d = \frac{2}{\mu_v} \rho_A r v \quad . \tag{A2. 2}$$

In Figure A2. 2 the Reynolds number R_d for a fireball of radius r in cm moving at a speed v in cm sec^{-1} in the atmosphere of viscosity $\mu_v \simeq 1.8 \cdot 10^{-4}$ g cm sec^{-1} is plotted as a function of height.

The plot for the drag coefficient has been taken from: Hoerner, S. F., "Fluid Dynamic Drag," published by the author, 1958, Section 3-8.

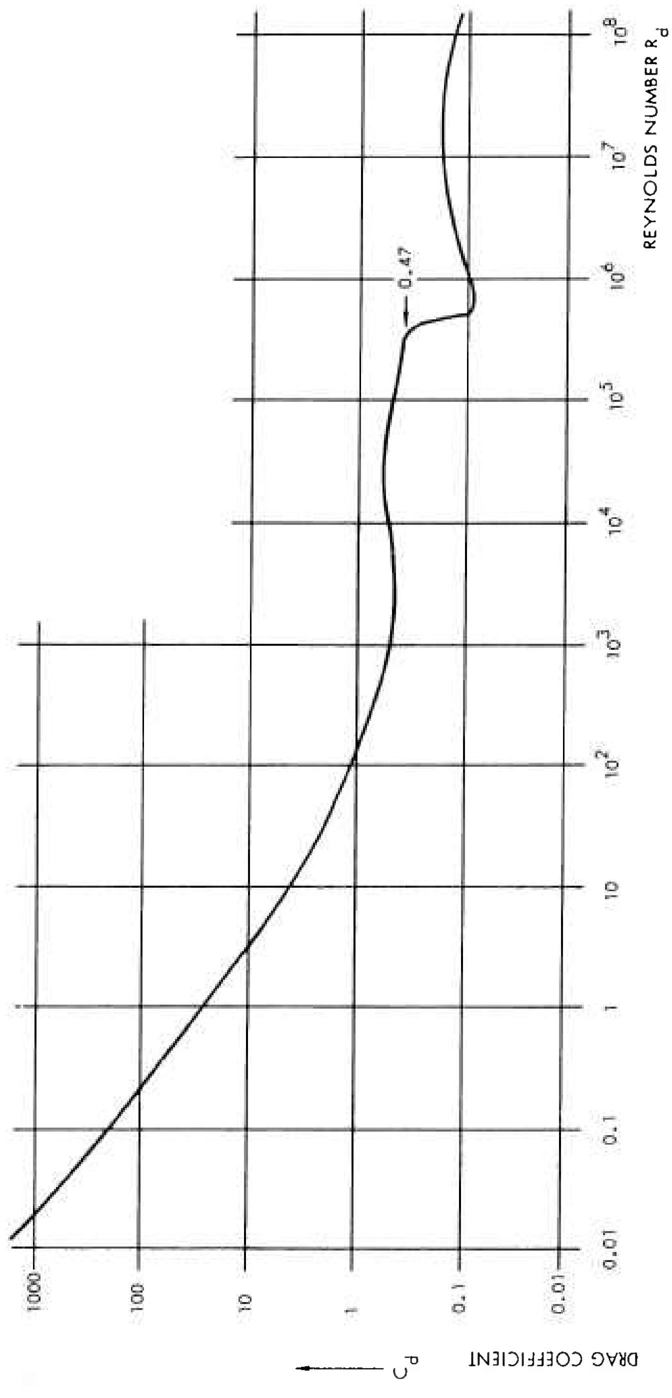

Figure A2. 1. Subsonic Drag Coefficient

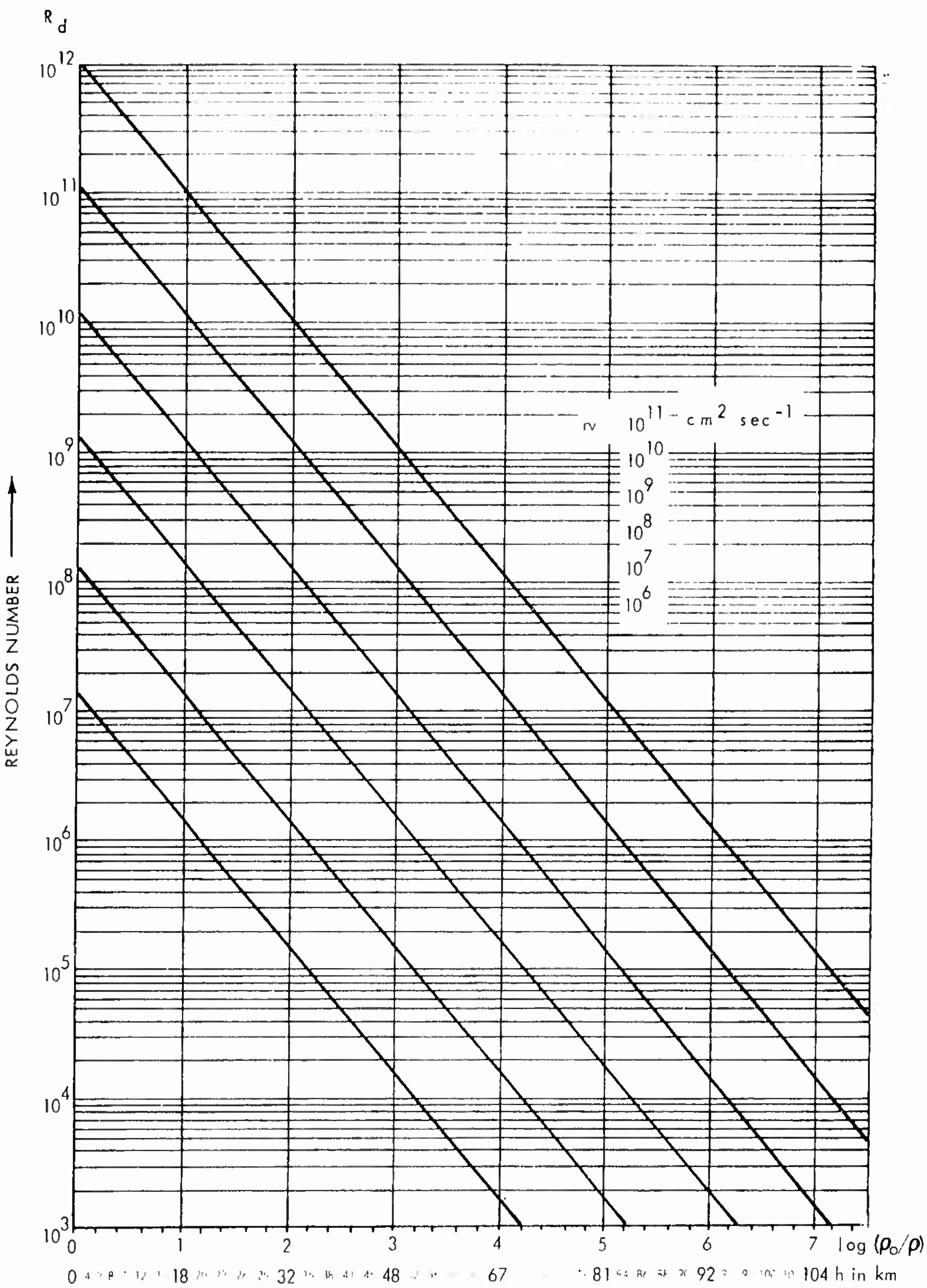

Figure A2. 2. Reynolds Number as a Function of Altitude for Various Fireballs

APPENDIX 3

THE DEFINITION OF AN AVERAGE FIREBALL TEMPERATURE*

Average fireball temperature is defined as that temperature which must prevail in the mixture of gases within the fireball in order to yield the total translational molecular energy of all its various components at their respective temperatures under the assumption of pressure equilibrium. To illustrate this formally let the subscripts $_O$ and $_{F,1}$ refer to the hot fireball gas, $_e$ to the entrained air and $_A$ to ambient air. R is the specific, $\Re$ the universal gas constant, M the mass of gas present, and μ the molecular weight. The definition of average fireball density $\overline{\rho}_F$ as the ratio of total mass to total volume gives

$$\overline{\rho}_F = \frac{M_O + M_e}{\dfrac{M_O}{\rho_{F,1}} + \dfrac{M_e}{\rho_e}} \quad . \tag{A3. 1}$$

The equality of translational molecular energy in the mixture of fireball gases at the average fireball temperature $\overline{T}_F$ and in the mixture of fireball gases at their respective temperatures leads from

$$\left(\frac{M_O}{\mu_{F,1}} + \frac{M_e}{\mu_e}\right) \overline{T}_F \, \Re = \left(\frac{M_O}{\mu_{F,1}} T_{F,1} + \frac{M_e}{\mu_e}\right) T_e \, \Re \tag{A3. 2}$$

to

$$\overline{T}_F = \frac{M_O R_{F,1} T_{F,1} + M_e R_e T_e}{M_O R_{F,1} + M_e R_e} \quad . \tag{A3. 3}$$

The conditions of pressure equilibirum

$$\rho_{F,1} R_{F,1} T_{F,1} = \rho_e R_e T_e = \rho_A R_A T_A \tag{A3. 4}$$

and

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* See Page 49.

$$\bar{\rho}_F \, \bar{R}_F \, \bar{T}_F = \rho_A \, R_A \, T_A \tag{A3.5}$$

lead to the average fireball gas constant

$$\bar{R}_F = \frac{M_0}{M_0 + M_e} R_{F,i} + \left(1 - \frac{M_0}{M_0 + M_e}\right) R_e \quad . \tag{A3.6}$$

The specific gas constant $R_{F,i}$ and the temperature $T_{F,i}$ of the originally hot fireball gas can be obtained from the tables on thermodynamic properties of air mentioned in Appendix 4. One must follow the isotropic changes of state of the originally hot fireball gas. The value of the gas constant and of the temperature can be read at that point where the entropy is equal to its initial value and the pressure equal to the ambient pressure. The ratio of initial mass M_0 to total mass $(M_0 + M_e)$ can be obtained from Equation (10.3) of the main body of this report.

APPENDIX 4

DEFINITION OF γ FOR A NON-IDEAL GAS AND γ FOR A MIXTURE OF GASES

Introduction

The internal energy of an ideal gas can be expressed by its state variables and a constant parameter γ. The same γ also appears as an exponent to the state variables in the adiabatic equation of state. For a real gas the internal energy can be expressed by its variables of state and a parameter γ in complete formal analogy to the description of an ideal gas. This γ is not a constant for real gases but depends upon their state variables. The adiabatic equation of state for real gases can only, for small changes of the state variables, be expressed in formal analogy to the corresponding law for an ideal gas. Even then the parameter γ appearing in the exponent to the state variables in the expression for the adiabatic equation of state is not identical with the parameter γ appearing in the expression for the internal energy. In addition, the adiabatic equation of state for a real gas loses all formal similarily to that for an ideal gas if large changes of the state variables are considered. In the following this will be discussed in detail.

On The Definition of γ For Ideal and Non-Ideal Gases

The internal energy $\mathcal{E}$ per mole of ideal gas at temperature T is

$$\mathcal{E} = \frac{\Re T}{\gamma - 1} \quad . \tag{A4. 1}$$

The universal gas constant is denoted by $\Re$. The internal energy per unit mass is obtained by dividing the internal energy per mole $\mathcal{E}$ by the molecular weight μ

$$E = \frac{\Re T}{\mu(\gamma - 1)} = \frac{RT}{\gamma - 1} \quad . \tag{A4. 2}$$

The specific gas constant is denoted by R. It is easy to see from Equation (A4.1) that γ is equal to the ratio of molecular specific heat at constant pressure and the molecular specific heat at constant volume if one keeps in mind that the universal gas constant is the difference between these two magnitudes.

The equation of state for an adiabatic process, that is for a process in which no energy in the form of heat is added to or removed from the volume of gas considered, is obtained from the first law of thermodynamics in the form

$$dE + p\,dV = 0 \quad . \tag{A4.3}$$

This relationship states that the sum of the change in internal energy dE per unit mass and of the external work vanish. The latter is the product of pressure p and change of specific volume V. The differential equation for an adiabatic change of state of an ideal gas is then obtained by substitution from Equation (A4.2) into Equation (A4.3) and by substituting for p and dp from the gas law

$$p = R\rho T \quad . \tag{A4.4}$$

The density is as usual referred to by ρ. In this way the differential equation for an adiabatic change of state of an ideal gas is obtained as

$$d\rho = \frac{\rho}{T}\,\frac{1}{\gamma-1}\,dT \quad . \tag{A4.5}$$

Integration and substitution for ρ and T from the ideal gas law (A4.4) yields

$$T\rho^{1-\gamma} = \text{const.}, \qquad Tp^{\frac{1-\gamma}{\gamma}} = \text{const.}, \qquad p\rho^{-\gamma} = \text{const.} \tag{A4.6}$$

The γ in Equation (A4.6) is identical with the γ in Equations (A4.1) and (A4.2).

For an ideal gas, γ or the ratio of specific heats is independent of the state variables and the internal energy varies linearly with temperature. The number of particles per mole of originally present ideal gas does not change as a function of the state variables.

What has been said about the properties of an ideal gas is approximately true for a real gas provided it is sufficiently far from its critical point. This implies that the gas must be at a temperature well above the critical temperature and at a density well below the critical density. The main constituents of air, N_2 and O_2 can be described by the γ of an ideal diatomic gas of 1.4 for temperatures at and above room temperature and for densities at or below normal. With increasing temperature any real gas eventually deviates from its ideal behavior. Heat energy begins to be transferred into internal molecular energy. At high enough temperatures the gas particles will become dissociated and ionized and thus the number of particles per mole of originally present gas will increase. As a consequence the internal energy does not vary linearly with temperature. The change in the number of particles present changes the molecular weight and therefore the specific gas constant. It is nevertheless useful to retain the definition of the internal energy of a gas per mole as

$$\mathcal{E} = \frac{\Re T}{\gamma(T, \rho) - 1} \quad . \tag{A4. 7}$$

and of the internal energy per unit mass as

$$E = \frac{R_{F,1}(T, \rho) T}{\gamma(T, \rho) - 1} \quad . \tag{A4. 8}$$

The function γ, now dependent upon the state variables, cannot any more be interpreted as the ratio of specific heats. Similarly the specific gas constant $R_{F,1}$ is a function of the state variables.

In Hilsenrath, T., et al., "Tables of Thermodynamic Properties of Air Including Dissociation and Ionization from 1500°K to 15,000°K," Arnold Engineering Development Center, Report TR-59-20, December 1959, the internal energy E^* per mole of the original (that is under normal conditions present) air is given as a function of temperature and density. The internal energy E^* is normalized in the mentioned reference by dividing it by the product of universal gas constant and temperature. (The universal gas constant in the mentioned reference is designated by R rather than by $\Re$, as it is done in the present paper.) The number Z^* of moles of gas present relative to one mole at normal conditions is also listed as a function of temperature and density. This allows one to compute the specific gas constant $R_{F,1}$ at a temperature T and density ρ from the specific gas constant R at normal conditions from

$$R_{F,1} = Z^* R \quad . \tag{A4. 9}$$

At temperature T and density ρ the internal energy per mole is according to the definition of γ in Equation (A4. 7)

$$\mathcal{E} = \frac{E^*}{Z^*} = \frac{\Re T}{\gamma(T, \rho) - 1} \quad . \tag{A4. 10}$$

From that γ can be computed as

$$\gamma(T, \rho) = \frac{E^*/\Re T + Z^*}{E^*/\Re T} \quad . \tag{A4. 11}$$

Since the enthalpy H^* per mole of original gas is $H^* = E^* + Z^*\Re T$, Equation (A4. 11) reduces to

$$\gamma(T, \rho) = \frac{H^*/\Re T}{E^*/\Re T} \quad . \tag{A4. 12}$$

This γ has been computed and is given in graphical form in Figure A4. 1.

To obtain the differential equation for an adiabatic change of state of a real gas, the internal energy per unit mass of a real gas is expressed from Equations (A4. 8) and (A4. 9) as

$$E = \frac{Z^* R T}{\gamma(T, \rho) - 1} \quad . \tag{A4. 13}$$

The gas law assumes the form

$$p = Z^* R \rho T \quad . \tag{A4. 14}$$

The differential equation for an adiabatic change of state for a real gas is obtained in the same way as for an ideal gas and turns out to be

$$d\rho = \left(\frac{\dfrac{1}{(\gamma-1)} \dfrac{\partial\gamma}{\partial T} - \dfrac{1}{Z^*} \dfrac{\partial Z^*}{\partial T} - \dfrac{1}{T}}{-\dfrac{1}{\gamma-1} \dfrac{\partial\gamma}{\partial\rho} + \dfrac{1}{Z^*} \dfrac{\partial Z^*}{\partial\rho} - \dfrac{\gamma-1}{\rho}} \; \frac{T}{\rho} \right) \frac{\rho}{T}\, dT \quad . \tag{A4. 15}$$

For small changes in temperature and density the functions γ and Z^* can be considered as constant and the differential quotients be approximated by difference quotients. Comparison with Equation (A4. 5) shows that for small changes in temperature and density a $\gamma_{F,i}$ may be defined, which allows the use of the adiabatic equations of state (A4. 6) of an ideal in the case of a real gas. This is done by

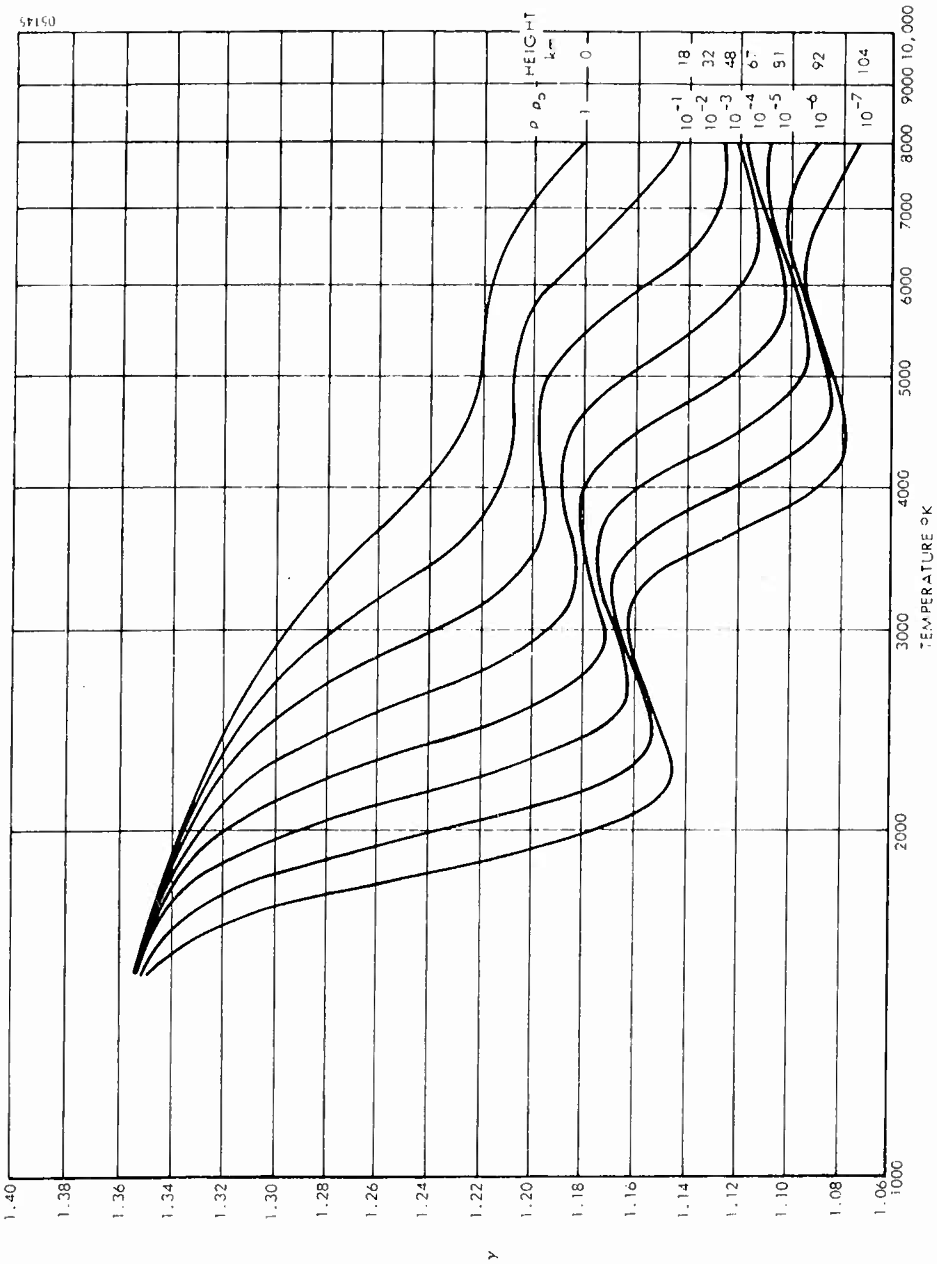

Figure A4. 1. γ Defining Internal Energy of Air as a Function of Temperature for Various Densities replacing γ, in the adiabatic equations of state (A4.6), by $\gamma_{F,1}$

$$\gamma_{F,1} = 1 + \frac{-\frac{1}{\gamma-1}\frac{\partial\gamma}{\partial\rho} + \frac{1}{Z^*}\frac{\partial Z^*}{\partial\rho} - \frac{\gamma-1}{\rho}}{\frac{1}{\gamma-1}\frac{\partial\gamma}{\partial T} - \frac{1}{Z^*}\frac{\partial Z^*}{\partial T} - \frac{1}{T}}\,\frac{\rho}{T} \quad . \tag{A4.16}$$

The $\gamma_{F,1}$ is different from the γ of Equations (A4.7), (A4.8), (A4.10), (A4.11) and (A4.12). It has been computed and is given in graphical form in Figure A4.2. For large changes in the state variables, Equation (A4.15) must be integrated. The result cannot be written in a form analogous to the adiabatic equation of state of an ideal gas.

The γ for a Mixture of Gases

It is of interest to express the properties of a mixture of gases in pressure equilibrium by its average temperature and average density. The average temperature may be defined as that temperature at which the translational molecular energy in the mixture of gas equals the total translational molecular energy of all its components at their respective temperatures. The average density is defined as the ratio of total mass to total volume. The $\overline{\gamma}_F$ which must be used in the adiabatic equation of state for a mixture of ideal gases in pressure equilibrium is obtained by the following considerations. The average density $\overline{\rho}$ is

$$\overline{\rho} = \frac{\sum_n M_n}{\sum_n (M_n/\rho_n)} \quad . \tag{A4.17}$$

The M_n refers to the mass, the ρ_n to the density of the n^{th} component. By differentiation

$$d\overline{\rho} = \overline{\rho}\,\frac{\sum_n (M_n/\rho_n)(d\rho_n/\rho_n)}{\sum_n (M_n/\rho_n)} \tag{A4.18}$$

is obtained. By expressing the adiabatic change of state of every component according to Equation (A4.5), Equation (A4.18) becomes

$$d\overline{\rho} = \frac{\overline{\rho}}{\sum_n (M_n/\rho_n)} \sum_n \frac{M_n}{\rho_n}\,\frac{1}{\gamma_n - 1}\,\frac{dT_n}{T_n} \quad . \tag{A4.19}$$

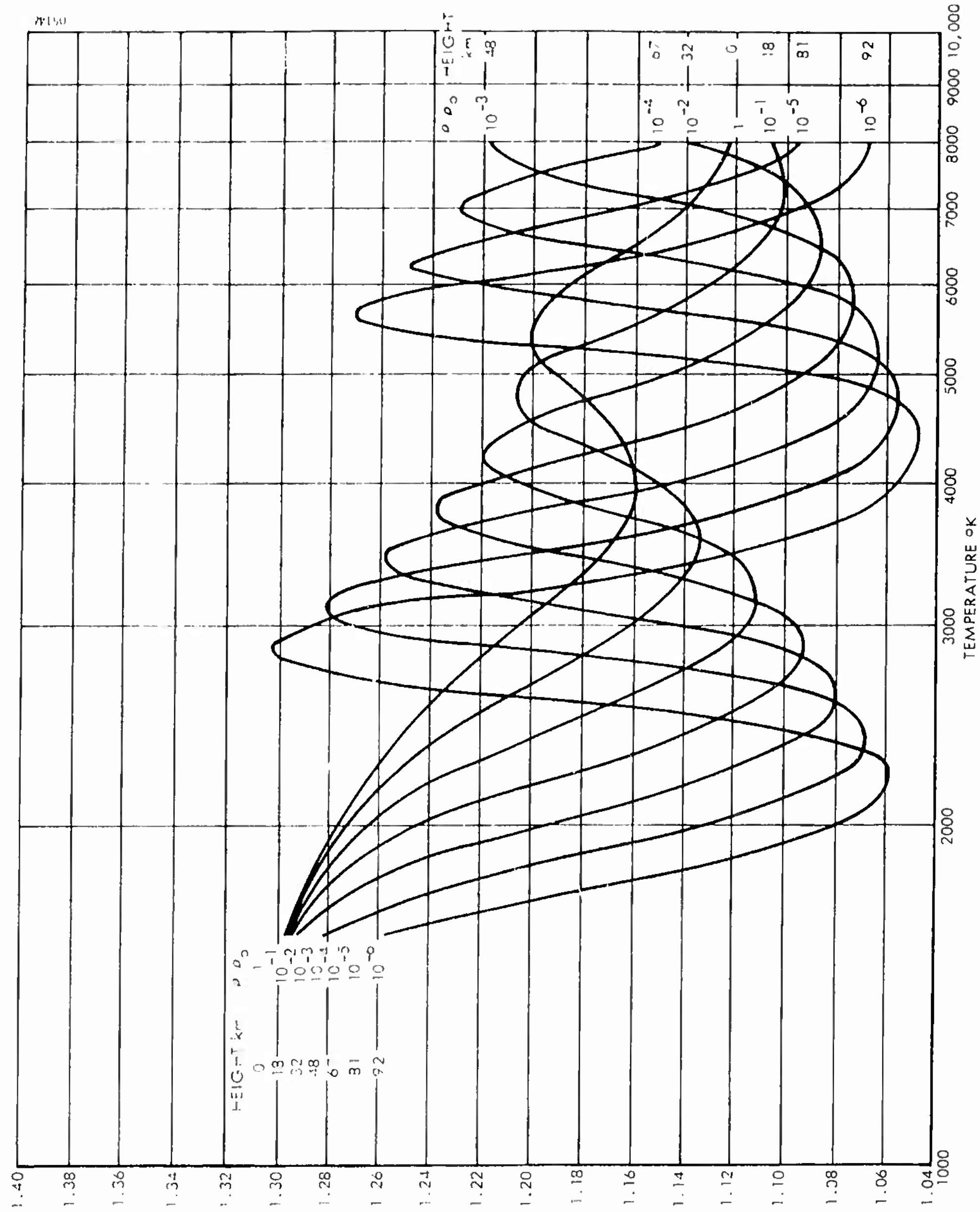

Figure A4. 2. $\gamma_{F,1}$ Defining Adiabatic Changes of State as a Function of Temperature for Various Densities

The same change in average density and temperature is experienced by a hypothetical ideal gas with a γ of $\overline{\gamma}_F$

$$d\overline{\rho} = \frac{\overline{\rho}}{\overline{T}} \frac{d\overline{T}}{\overline{\gamma}_F - 1} \quad . \tag{A4.20}$$

Substitution of the adiabatic change in pressure corresponding to the adiabatic change in temperature

$$-\frac{\gamma_n - 1}{\gamma_n} \frac{dp_n}{p_n} = \frac{dT_n}{T_n} \quad \text{and} \quad -\frac{\overline{\gamma}_F - 1}{\overline{\gamma}_F} \frac{dp}{p} \frac{d\overline{T}}{\overline{T}} \tag{A4.21}$$

transforms Equations (A4.19) and (A4.20) into

$$d\overline{\rho} = -\frac{\overline{\rho}}{\sum\limits_n (M_n/\rho_n)} \sum_n \frac{M_n}{\rho_n} \frac{1}{\gamma_n} \frac{dp_n}{p_n} = -\frac{\overline{\rho}}{\overline{\gamma}_F} \frac{dp}{p} \quad . \tag{A4.22}$$

The condition of pressure equilibrium between all the components allows to write Equation (A4.22) in the form

$$d\overline{\rho} = -\frac{\overline{\rho}}{\sum\limits_n (M_n/\rho_n)} \frac{dp}{p} \sum_n \frac{M_n}{\rho_n} \frac{1}{\gamma_n} = -\frac{\overline{\rho}}{\overline{\gamma}_F} \frac{dp}{p} \quad . \tag{A4.23}$$

With

$$\sum_n (M_n/\rho_n) = \sum_n (M_n/\overline{\rho}) \quad , \tag{A4.24}$$

Equation (A4.23) assumes the form

$$d\overline{\rho} = -\frac{\overline{\rho}^2}{\sum\limits_n M_n} \sum_n \frac{M_n}{\rho_n} \frac{1}{\gamma_n} = -\frac{\overline{\rho}}{\overline{\gamma}_F} \quad . \tag{A4.25}$$

In this way

$$\overline{\gamma}_F = \frac{1}{\dfrac{\overline{\rho}}{\sum\limits_n M_n} \sum\limits_n \dfrac{M_n}{\rho_n} \dfrac{1}{\gamma_n}} \tag{A4.26}$$

is obtained. It is interesting that this is an average using the volumes as the weight factors. This $\overline{\gamma}_F$ must be used in the adiabatic equation of state to express average temperature and average density of a mixture of ideal gases in pressure equilibrium. It is seen that not only the masses M_n of the various components and their respective ν_n have to be known, but also their densities ρ_n have to be known so that $\overline{\nu}_F$ may be computed. Since the ρ_n change during an adiabatic process, the definition of an average $\overline{\nu}_F$ is only good for small changes of the state variables. In case the mixture of gas contains real gases the dorresponding ν_n's must be replaced by the appropriate $\nu_{F,i}$'s.

APPENDIX 5

SCALE HEIGHTS

The introduction of a scale height H_{sT} of temperature requires some explanation. Usually the variation of atmospheric temperature with altitude is not described by scale heights. For large ranges in altitude such a description would be inconvenient because the atmospheric temperature varies linearly with altitude. For small changes in altitude one obtains for a temperature gradient dT/dh

$$T - \frac{dT}{dh}\Delta h = T\,e^{-\frac{\Delta h}{H_{sT}}} = T\left(1 - \frac{\Delta h}{H_{sT}}\right) , \qquad (A5.1)$$

and the local scale height is

$$H_{sT} = \frac{T}{dT/dh} . \qquad (A5.2)$$

One sees that a description in terms of a scale height H_{sT} is valid as long as the altitude differences considered are small compared to H_{sT}. This is always the case for calculations on fireball rise. Hence the introduction of a scale height of temperature seems justified. It is also seen that a positive scale height implies a decrease of temperature with altitude, while a negative scale height describes an increase.

Table A5.1 lists the scale heights of atmospheric density $H_{s\rho}$ and atmospheric temperature H_{sT}. The equivalent scale height H_{se} given in Equation (5.10) and the ratio $H_{s\rho}/H_{se}$ are also tabulated.

Table A5.1. Scale Heights as a Function of Altitude

Altitude (km)	$H_{s\rho}$ (km)	H_{sT} (km)	H_{se} (km)	$H_{s\rho}/H_{se}$ (km)
0				
	7.80	+44	6.63	1.03
5				
	7.31	+39	6.14	1.17
11				
	6.87	∞	6.87	1.00
15				
	6.54	∞	6.54	1.00
20				
	6.40	∞	6.40	1.00
25				
	6.57	−73	7.25	0.906
30				
	7.10	−79	7.80	0.910
35				
	7.46	−85	8.19	0.911
40				
	8.10	−91	9.00	0.900
47				
	8.36	∞	8.36	1.00
50				
	8.42	∞	8.42	1.00
53				
	8.0	+60	7.04	1.12
60				
	7.34	+55	6.50	1.13
65				
	6.7	+50	5.92	1.13
70				
	5.92	+46	5.24	1.11
75				
	5.37	+42	4.97	1.08
79				
	4.98	∞	4.98	1.00
85				
	4.98	∞	4.98	1.00
90				
	5.16	−50	6.25	0.826

(continued)

Table A5. 1. (continued)

Altitude (km)	$H_{s\rho}$ (km)	H_{sT} (km)	H_{se} (km)	$H_{s\rho}/H_{se}$ (km)
	5. 16	-50	6. 25	0. 826
95				
	5. 59	-55	6. 21	0. 900
100				
	6. 22	-60	6. 94	0. 896
105				
	6. 83	-15	12. 5	0. 546
110				
	10. 4	-20	21. 8	0. 477
115				
	12. 5	-25	25. 0	0. 500
120				
	16. 1	-30	34. 5	0. 467
125				
	18. 3	-35	38. 4	0. 477
130				
	21. 9	-40	50. 0	0. 438
135				
	24. 0	-45	50. 0	0. 480
140				
	27. 7	-50	62. 5	0. 443
145				
	30. 3	-55	66. 7	0. 454

APPENDIX 6

HEAT EXCHANGE BETWEEN THE HOT FIREBALL GAS AND THE ENTRAINED AIR

In the main body of this report the assumption has been made that no heat is exchanged between the hot fireball gas and the entrained air. The calculation of the various physical quantities of interest has then been carried out by using an average temperature. The true temperature of the fireball gases are those obtained by adiabatic expansion from their initial values. The average temperature is defined in Appendix 3. It is not the temperature which would be achieved after fireball gas and entrained air have exchanged heat; it is that temperature which must prevail in the mixture of hot fireball gas and the entrained air, which have not exchanged any heat, so that pressure equilibrium is maintained between the fireball and the ambient atmosphere.

The average temperature so defined would be the temperature after heat exchange, if both fireball gas and entrained air did have the same γ in the expression for the internal energy per unit mass (Equation (A4. 2) of Appendix 4). This is not the case because the fireball gas is ambient air which has been heated to a high temperature and in the process has deviated from its ideal behavior. Dissociation and ionization have taken place and the internal energy of the hot gas is much larger than one would expect from its ideal behavior at low temperatures. If therefore hot fireball gas and entrained air exchange heat the resulting temperature would be higher than the average temperature mentioned above. As a consequence the fireball density would be lower, the fireball radius larger.

It is easy to lift the assumption that no heat exchange takes place between the hot fireball gas and the entrained air. The computation of fireball rise becomes somewhat more complicated in the process and insight into the phenomena taking place is somewhat obscured. The computation remains similar to that presented in the main body of this report. The details are as follows.

The calculation starts by assuming a certain initial ratio $(\overline{\rho}_F/\rho_A)_0$ of fireball density to ambient air density. From this and Equation (10. 3) of the main body of this report the mass M_1 of the fireball at the end of the first step can be computed. The first step is the time it takes the fireball to rise a height equal to its initial radius. At the end of the step the hot fireball gas and the entrained air are assumed to completely exchange heat. The process occurs at constant pressure. For this reason the thermodynamic potential which is conserved is the enthalpy. This can be expressed by

$$H_{F,1} = \frac{M_0}{M_1} H_{F,0} + \left(1 - \frac{M_0}{M_1}\right) H_A \quad . \tag{A6. 1}$$

The enthalpy per unit mass is denoted by H, the mass of air by M. The subscript $_{F,0}$ refers to the initial fireball mass, $_{F,1}$ to the fireball mass at the end of the first step and $_A$ to the ambient air.

It is therefore easy to compute the enthalpy per unit mass $H_{F,1}$ after heat exchange. It is also known at what pressure the fireball is at the end of the first step; this pressure is the ambient pressure encountered at a height equal to the initial fireball radius above the initial height of the fireball center. The two variables of state, enthalpy and pressure, can be used to determine temperature and density of the fireball gas after mixing by using tables of the thermodynamic properties of air.

The density so found can be used to compute the fireball radius after heat exchange. It also sets the initial condition for the ratio of average fireball density to ambient air density in the calculation of the second step. In this way buoyant fireball rise can be computed step by step, allowing complete heat exchange to occur. The transition to expansion controlled rise is determined from inequality (8. 3) of the main body of this report, by using the values of fireball density and fireball radius obtained by taking heat exchange into account.

Consideration of the case where incomplete or delayed exchange of heat is assumed proceeds in an analogous manner. The details are left to the reader.

T. Hilsenrath's, et al., tables of thermodynamic properties of air quoted in Appendix 4 are very useful for the present purpose. From these tables the γ for the determination of internal energy, the $\gamma_{F,1} = \overline{\gamma}$ for the adiabatic equation of state, the sound speed C and the enthalpy H per unit mass have been computed. The tables themselves, in addition to the magnitudes mentioned, have been listed and can be supplied upon request.

REFERENCES

1. Rohringer, G. Properties of Ballistic Fireball Rise. (RM 63TMP-26) Santa Barbara, Calif.: G. E. TEMPO, 1 July 1963. UNCLASSIFIED.